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MODULE 1

MATHEMATICS 30

Polynomial Functions

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Mathematics 30

Module 1

POLYNOMIAL FUNCTIONS

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Student Module Booklet
Module 1
Polynomial Functions
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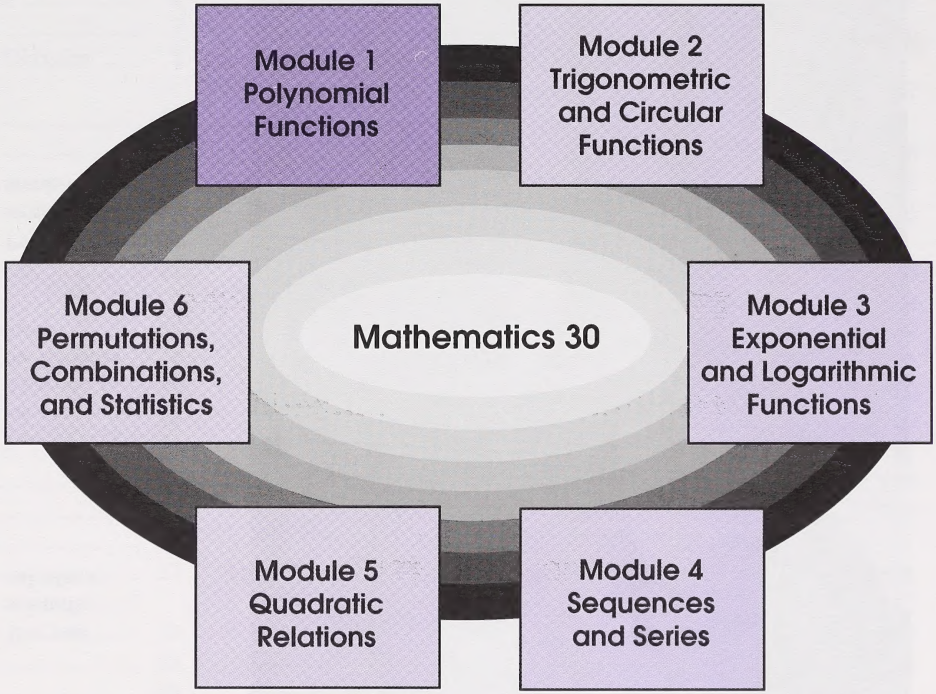
Welcome

WESTFILE INC.

Welcome to Module 1. We hope you'll enjoy your study of Polynomial Functions.

Mathematics 30 contains six modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.



The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

- Use your graphing calculator.

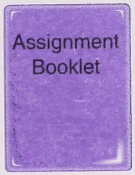
- Use your scientific calculator.

- Use computer software.

- Use the suggested answers in the Appendix to correct the activities.

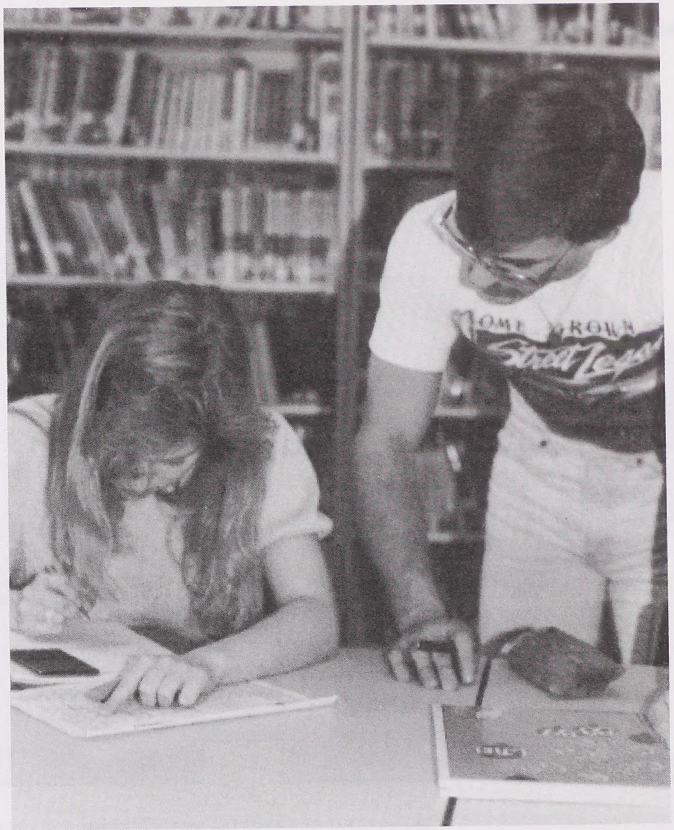
- View a videocassette.

- Pay close attention to important words or ideas.



- Answer the questions in the assignment booklet.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.



There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

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Module Overview

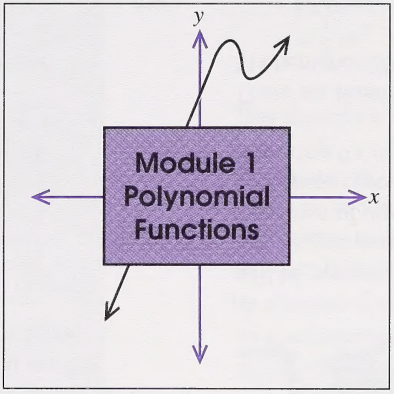
It is a beautiful day for skiing at Lake Louise. A group of skiers decide to comb the ski slopes in search of a good jump. To add a little excitement, they wish to have a little competition amongst themselves. The distance a skier jumps depends on the skier's speed, acceleration, and angle of jump.

Do you know how factors like speed, acceleration, and length of jump are related? These variables can be related using a polynomial function. By evaluating the function, by solving the polynomial equation, or by graphing the polynomial function, you can determine the distance the skier will travel.

In this module you will discover what a polynomial function is. You will examine the terminology and the properties of these polynomial functions. You will be shown how to factor, solve equations, sketch graphs, and determine polynomial expressions from given values.

Section 1
Introduction to Polynomial Functions

Section 2
Remainder Theorem and Factor Theorem



Section 3
Solving Polynomial Equations

Section 4
Graphing Polynomial Functions

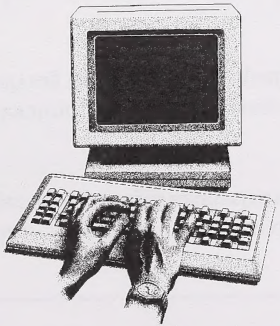
Evaluation

Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete four section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	**8 marks**
Section 2 Assignment	**18 marks**
Section 3 Assignment	**26 marks**
Section 4 Assignment	**18 marks**
Final Module Assignment	**30 marks**
TOTAL	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.

If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Diploma Exam Preparation

Since you will be writing a Mathematics 30 Diploma Examination when you have finished this course, it is important that you are familiar with the types of questions that will be asked and the format in which they will be presented. From time to time, throughout the modules in this course, you may be given several activity questions and assignment questions of diploma exam format so that you can become familiar with these types of questions. For more detailed information on preparing for diploma exams, consult the Appendix in this module.

There are three basic types of questions you will encounter on a Mathematics 30 Diploma Exam:

- **Multiple Choice:** With each question you are given four choices, listed by A., B., C., and D., from which you choose the single **best** possible answer. Each multiple choice question is worth one mark. You record your choice for each question by letter on a separate answer sheet provided.

- **Numerical Response:** A numerical response question is designed so that you must record your answer on a machine-scored answer sheet.

- **Written Response:** Written response questions are multiple-mark questions in which you are expected to show all your work and organize and present your answer in a clear and precise manner. Marks may be given for clarity in the communication of your answer as well as the shown calculation work.

Section 1: Introduction to Polynomial Functions

After walking for several hours, a tourist stops and rests on the Golden Gate Bridge in San Francisco, California. The view of the water below is great. Curious to know how high the bridge is, the tourist drops a stone over the edge and waits to see the splash. How can the tourist use the time it takes the stone to reach the water to calculate the height of the bridge?

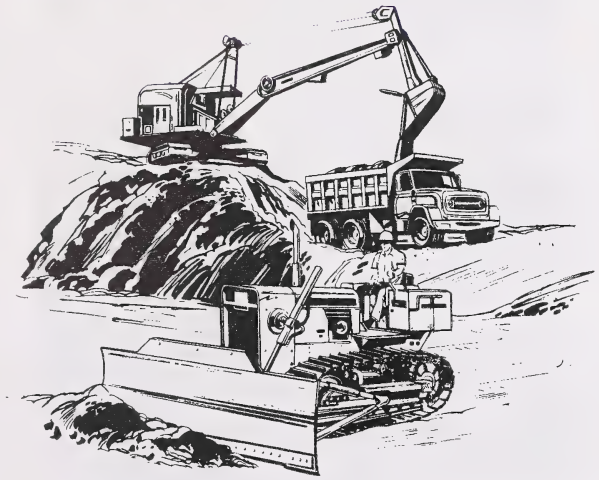
To answer this question, a formula relating distance and time is needed. The formula is as follows:

$$d = \frac{1}{2}at^2$$

This formula is an example of a polynomial function.

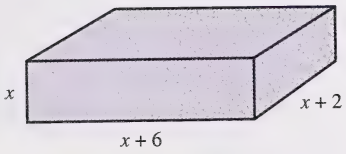
In this section you will be shown how to recognize polynomial functions and give examples and nonexamples. You will also be shown how to classify polynomial functions. Finally, you will discover how to divide integral polynomial functions in one variable by a binomial of the form $(x-a)$ and write the division operation on a polynomial function by a binomial in the form of the Division Algorithm: $P(x) = D(x)Q(x) + R$.

Activity 1: Recognizing a Polynomial Function



The volumes of the boxes of gravel trucks vary according to their dimensions. For example, a box could have a length 6 m more than its height, and a width 2 m more than its height.

The volume of this box is $x(x+6)(x+2)$ or x^3+8x^2+12x cubic units.



The volume can be written as a function of x.

$$f(x)=x^3+8x^2+12x \text{ cubic units}$$

What kind of function is this? It is a **polynomial function.**

Recall that a function is a relation in which one quantity depends on another.

A polynomial in a single variable x is an expression of the form $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^1 + a_0$, where $a_n \neq 0$.

- The numerical coefficients a_n, a_{n-1}, ..., a_0 are real numbers. a_n is called the initial or leading coefficient, and a_0 is the constant term. The terms are written in descending degree.

- The exponents n, $n-1$, ... are positive integers.

Therefore, a polynomial function has this form.

$$f(x)=a_n x^n + a_{n-1}x^{x-1} + \cdots + a_0, \text{ where } a_n \neq 0$$

The polynomial function is of degree n.

$f(x)$, y, and $P(x)$ all mean the same.

In this activity you will use the terminology for polynomial functions and identify polynomial functions.

Example 1

For the polynomial function $y = 5x^4 + 7x^3 - 4x^2 - 8x + 6$, find the following:

- the numerical coefficients
- the degree
- the constant term
- the initial coefficient

Solution

The numerical coefficients of $y = 5x^4 + 7x^3 - 4x^2 - 8x + 6$ are $a_4 = 5$, $a_3 = 7$, $a_2 = -4$, $a_1 = -8$, and $a_0 = 6$.

6 can be written as $6x^0$ (except when $x = 0$). Thus, 6 is the coefficient of x^0, where $x^0 = 1$.

Note: The subscripts on the coefficients are the same as on the exponents on the variable. For example, the subscript 4 on a_4 is the same as the exponent 4 on x^4.

Since the highest exponent of the variable is 4 in the term $5x^4$, the degree is 4.

The constant term in the function is 6.

The initial coefficient in the function is 5.

Example 2

Are the following expressions polynomial functions?

- $y = -5x^3 + x^{\frac{3}{2}} - 4$
- $y = 2x^2 - 7x^{-1} + 3$
- $y = 4x + \sqrt{8x}$

Solution

Since the exponent $\frac{3}{2}$ is not a positive integer, $y = -5x^3 + x^{\frac{3}{2}} - 4$ is not a polynomial function.

Since the exponent -1 is not a positive integer, $y = 2x^2 - 7x^{-1} + 3$ is not a polymonial function.

Since $\sqrt{8x} = \sqrt{8}x^{\frac{1}{2}}$ and does not have a positive integral exponent, $y = 4x + \sqrt{8x}$ is not a polynomial function.

Here is a summary for expressions that are **not** polynomials.

- expressions containing coefficients which are not real (for example, $\sqrt{-3x}$ and $\sqrt{-5}$)
- expressions containing negative or fractional exponents of variables (for example, x^{-1} and $x^{\frac{1}{5}}$)
- expressions containing roots of variables (for example, $\sqrt[3]{x}$ and $\sqrt{x}$)

$\sqrt[3]{x} = x^{\frac{1}{3}}$

$\sqrt{x} = x^{\frac{1}{2}}$

- expressions containing variables in the denominator which have positive integral exponents (for example, $\frac{8}{x}$ and $\frac{7}{x^2}$)

$\frac{8}{x} = 8x^{-1}$

$\frac{7}{x^2} = 7x^{-2}$

1. For each polynomial function, find the following:

- the numerical coefficients
- the degree
- the constant term
- the initial coefficient

a. $y = 2x^4 + 4x^2 + x - 8$

b. $y = x^5 - 2x^4 - x + 2$

c. $y = 8$

2. Are the following expressions real polynomials?

a. $\frac{3}{x^2} - 4x$

b. 8

c. $\frac{3}{4}x^4 - \frac{1}{5}x - \frac{2}{3}$

d. $2x^3 - \sqrt[5]{x^2} + \frac{1}{2}$

e. $\frac{5}{x^{-2}} + 4$

f. $7 - 3x^{-2} + x^4$

g. $x^2 - 4x^3 + \sqrt{-3}$

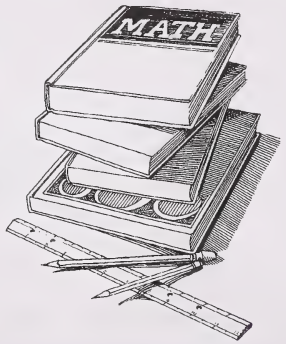
h. $\sqrt{7}x^3 - \frac{1}{2}x$

i. $-4x^4 - \sqrt{-8}x^2 + \frac{1}{4}$

j. $5x^5 - \frac{4}{x} + 3$

Check your answers by turning to the Appendix.

Now you know how to recognize a polynomial function. This should help you to classify polynomial functions.



Activity 2: Classifying Polynomial Functions

In previous mathematics courses, you have learned to classify polynomials according to the number of terms.

Number of Terms	Classification
one	monomial
two	binomial
three	trinomial

Note: When polynomials contain more than three terms, they are simply called polynomials.

Polynomial functions can be classified according to degree or according to the number of terms.

Classifying Polynomial Functions According to Degree

The term with the highest degree in a polynomial function determines the degree of that polynomial function.

For example, the degree of $y = 3x^4 + x^3 - 8x^2 + 7x - 5$ is 4 since the term $3x^4$ has a degree of 4.

Examine the following table of different polynomial functions showing their degree and another common name for the polynomial functions.

Polynomial Function	Degree	Common Name
$y = 5$	0	constant function
$y = 2x - 1$	1	linear function
$y = -3x^2 + 7x - 5$	2	quadratic function
$y = x^3 - 4x^2 + 9x + 3$	3	cubic function
$y = -2x^4 + 3x^3 - x^2 + 5x - 2$	4	quartic function
$y = x^5 - x^4 + 45x^2$	5	quintic function

Classifying Polynomial Functions According to Coefficients

If all the coefficients of a polynomial function are integers, then the function is an integral polynomial function.

A rational polynomial function has rational numbers for coefficients. A polynomial function, which has real numbers for coefficients, is a real polynomial function.

Note: An integral polynomial is also rational or real; but the most descriptive term is used.

The following table classifies polynomial functions according to coefficients.

Polynomial Function	Coefficient Classification	Reason
$y = 3x^2 - x + 5$	integral	3, −1, and 5 are integers.
$y = -2x^2 - \frac{1}{3}$	rational	$-\frac{1}{3}$ is a rational number.
$y = 5x^2 - \sqrt{2}x + 7$	real	$-\sqrt{2}$ is an irrational number that belongs to the real numbers.

Look at the next example, which puts these ideas into practice.

Example

Is the expression $y = -2x^2 + \frac{3}{2}x + \sqrt{5}$ a polynomial function? If it is a polynomial function, classify it according to degree, coefficients, and number of terms.

Solution

Yes, the expression is a polynomial function.

The degree is 2. Therefore, it is a quadratic function.

The coefficients are -2 (an integer), $\frac{3}{2}$ (a rational number), and $\sqrt{5}$ (an irrational number). These numbers belong to the set of real numbers. Therefore, the expression is a real polynomial function.

Classify each polynomial function according to degree and coefficients.

1. $y = 2x + 3x^2$

2. $y = \frac{5}{2} + 4x^4 - x^3$

3. $y = 7x - 3$

4. $y = \sqrt{8}$

5. $y = 7x - 5x^2 + 3x^4 - 4$

6. $y = x^5 - 3x^4 - \frac{3}{2}x^2$

7. $y = 3x^2 - \sqrt{5}x$

8. $y = -3x^2$

9. $y = 2x^2 - \frac{1}{4}x^5 - 5$

10. $y = \sqrt{7}x^3$

Check your answers by turning to the Appendix.

In this activity you classified a polynomial according to its degree or coefficient of the terms.

Activity 3: Dividing Polynomials Using Long Division

From previous mathematics courses, you learned that one whole number is divisible by another whole number if the remainder is zero. Look at the following division problems.

$$\begin{array}{r} 4 \\ 4\overline{)16} \\ \underline{16} \\ 0 \end{array} \quad \leftarrow \text{remainder}$$

This is divisible since the remainder is zero.

$$\begin{array}{r} 2 \quad \leftarrow \text{quotient} \\ \text{divisor} \rightarrow 6\overline{)14} \leftarrow \text{dividend} \\ \underline{12} \\ 2 \quad \leftarrow \text{remainder} \end{array}$$

This is not divisible since the remainder is not zero.

This long division procedure can be used to check if one polynomial is a factor of a given polynomial. If the divisor is a factor of the dividend, then the remainder must be zero.

Example 1

Divide $3x^3 + 4x - 5$ by $x - 1$.

Solution

$$\begin{array}{r} 3x^2 + 3x + 7 \quad \leftarrow \text{quotient} \\ \text{divisor} \rightarrow x-1\overline{)3x^3 + 0x^2 + 4x - 5} \leftarrow \text{dividend} \\ \underline{3x^3 - 3x^2} \\ 3x^2 + 4x \\ \underline{3x^2 - 3x} \\ 7x - 5 \\ \underline{7x - 7} \\ 2 \quad \leftarrow \text{remainder} \end{array}$$

Follow these steps.

Step 1: Arrange the terms of the divisor and the dividend in descending order. The exponents of the variable must be decreasing from left to right.

Step 2: When terms in the dividend or divisor are missing, use a zero coefficient. In Example 1, the x^2-term is missing in the dividend; thus, it is replaced by $0x^2$.

Step 3: The x in the divisor must be multiplied by $3x^2$ in the quotient to produce the first term $(3x^3)$ in the dividend.

Step 4: Multiply $x - 1$ by $3x^2$ to get $3x^3 - 3x^2$. Now, subtract $3x^3 - 3x^2$ from $3x^3 + 0x^2$ to get $3x^2$. Bring down $+4x$ so it is beside $3x^2$; that is, $3x^2 + 4x$.

Step 5: The x in the divisor must be multiplied by $3x$ in the quotient to produce the first term $(3x^2)$ in $3x^2 + 4x$.

Step 6: Multiply $x - 1$ by $3x$ to get $3x^2 - 3x$. Now, subtract $3x^2 - 3x$ from $3x^2 + 4x$, which gives $7x$. Next, bring down -5 so that it is beside $7x$; that is $7x - 5$.

Step 7: The x in the divisor must be multiplied by the 7 in the quotient to produce the first term $(7x)$ in $7x - 5$.

Step 8: Multiply $x - 1$ by 7, which gives $7x - 7$. Then, subtract $7x - 7$ from $7x - 5$, which gives 2.

In Example 1, $3x^3+4x-5$ was divided by $x-1$.

The division may be shown in the following form.

$$\frac{3x^3+4x-5}{x-1}=3x^2+3x+7+\frac{2}{x-1}$$

$$3x^3+4x-5=(x-1)\left(3x^2+3x+7\right)+2$$

Multiply both sides of the equation by $x-1$.

This last form illustrates that the
dividend = (divisor) × (quotient) + remainder.

The remainder R is always a constant not containing x when the divisor is of the form $x-a$; that is, x is to the first power.

If the division method is correct, then the divisor times the quotient plus the remainder should equal the dividend.

$$\begin{array}{r} 3x^2+3x+7 \leftarrow \text{quotient} \\ \hline x-1 \leftarrow \text{divisor} \\ -3x^2-3x-7 \\ \hline 3x^3+3x^2+7x \\ \hline 3x^3+0x^2+4x-7 \\ \hline +2 \leftarrow \text{remainder} \\ \hline 3x^3+4x-5 \leftarrow \text{dividend} \end{array}$$

$$3x^3+4x-5=(x-1)\left(3x^2+3x+7\right)+2$$

dividend = (divisor) × (quotient) + remainder

Note: Since there is a remainder of 2, then $x-1$ is not a factor of $3x^3+4x-5$.

Example 2

Divide $2x^3-7x^2+7x-2$ by $x-2$.

Solution

$$\begin{array}{rl} & 2x^2-3x+1 \leftarrow \text{quotient } Q(x) \\ \text{divisor } D(x) \rightarrow & x-2\overline{)\,2x^3-7x^2+7x-2} \leftarrow \text{dividend } P(x) \\ & \underline{2x^3-4x^2} \\ & -3x^2+7x \\ & \underline{-3x^2+6x} \\ & x-2 \\ & \underline{x-2} \\ & 0 \leftarrow \text{remainder } R \end{array}$$

$$2x^3-7x^2+7x-2=(x-2)\left(2x^2-3x+1\right)+0$$

Note: Since there is a remainder of zero, then $x-2$ is a factor of $2x^3-7x^2+7x-2$.

1. Use long division to divide the following.

a. $(3x^3 - 2x^2 - 7) \div (x - 3)$

b. $(4x^2 - 2x^3 + 3x - 6) \div (x + 2)$

c. $(-5x - 8 + 6x^2) \div (2x - 1)$

d. $(8x^4 - 4x^2 + 3x - 5) \div (2x + 3)$

Check your answers by turning to the Appendix.

The same principle of divisibility also applies to polynomial functions. A polynomial function $P(x)$ is divisible by another polynomial function $D(x)$ if the remainder R is zero.

$P(x)$ is read P of x.
$D(x)$ is read D of x.

If $P(x)$ and $D(x)$ are polynomial functions and $D(x)$ is a nonconstant polynomial function, then there exist polynomials $Q(x)$ and $R(x)$ such that $P(x) = [D(x) \times Q(x)] + R(x)$, where either $R(x) = 0$ or the degree of R is less than the degree of $D(x)$.

The polynomials $P(x)$, $D(x)$, $Q(x)$, and $R(x)$ are the dividend, the divisor, the quotient, and the remainder respectively.

Note: Whenever you divide a polynomial by a first-degree binomial, the remainder will be a constant. In this case, it is acceptable to use R instead of $R(x)$ for the remainder.

$D(x)$ is a factor of $P(x)$ when $R = 0$.

Example 3

If $P(x) = x^4 - x^3 - 8x^2 + 3$ and $D(x) = x - 3$, determine $Q(x)$ and $R(x)$; then verify that $P(x) = [D(x) \times Q(x)] + R(x)$. Is $D(x)$ a factor of $P(x)$?

Solution

$$
\begin{array}{r}
x^3 + 2x^2 - 2x - 6 \\
x - 3 \overline{)\, x^4 - x^3 - 8x^2 + 0x + 3} \\
\underline{x^4 - 3x^3} \\
2x^3 - 8x^2 \\
\underline{2x^3 - 6x^2} \\
-2x^2 + 0x \\
\underline{-2x^2 + 6x} \\
-6x + 3 \\
\underline{-6x + 18} \\
-15
\end{array}
$$

$Q(x) = x^3 + 2x^2 - 2x - 6$

$R = -15$

$$
\begin{array}{rl}
x^3 + 2x^2 - 2x - 6 & \leftarrow Q(x) \\
\hline
x - 3 & \leftarrow D(x) \\
\hline
-3x^3 - 6x^2 + 6x + 18 & \\
x^4 + 2x^3 - 2x^2 - 6x & \\
\hline
x^4 - x^3 - 8x^2 + 0x + 18 & \\
-15 & \leftarrow R(x) \\
\hline
x^4 - x^3 - 8x^2 + 3 & \leftarrow P(x)
\end{array}
$$

Thus, $P(x) = [D(x) \times Q(x)] + R(x)$.

Since R is not equal to zero, $D(x)$ is not a factor of $P(x)$.

2. $P(x) = 4x^3 - 2x^2 - 6$, and $D(x) = x - 2$.

a. Determine $Q(x)$ and R.

b. Verify that $P(x) = [D(x) \times Q(x)] + R(x)$.

c. Is $D(x)$ a factor of $P(x)$?

3. $P(x) = 3x^3 - 6x + 4$, and $D(x) = x + 1$.

a. Determine $Q(x)$ and R.

b. Verify that $P(x) = [D(x) \times Q(x)] + R(x)$.

c. Is $D(x)$ a factor of $P(x)$?

4. $P(x) = 2x^4 - 5x + 3$, and $D(x) = x - 1$.

a. Determine $Q(x)$ and R.

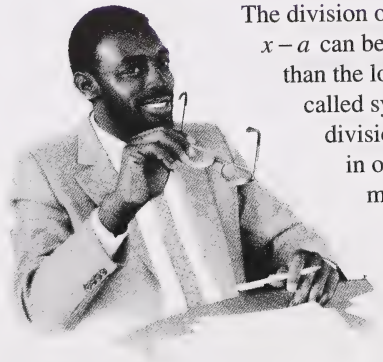
b. Verify that $P(x) = [D(x) \times Q(x)] + R(x)$.

c. Is $D(x)$ a factor of $P(x)$?

Check your answers by turning to the Appendix.

You should have a good grasp of this activity before trying the synthetic division in the following activity.

Activity 4: Dividing Polynomials Using Synthetic Division



The division of an integral polynomial by $x - a$ can be achieved by a simpler method than the long division method. This is called synthetic division. Synthetic division is division of a polynomial in one variable, say x divided by x minus a constant, making use of detached coefficients and a simplified arrangement of the work. In the next example you will see how synthetic division is developed.

Example 1

Divide the polynomial $2x^4 - 7x^3 + 5x + 4$ by $x - 2$.

Solution

Method 1: Long Division

$$
\begin{array}{r}
2x^3 - 3x^2 - 6x - 7 \\
x - 2 \overline{)\,2x^4 - 7x^3 + 0x^2 + 5x + 4} \\
\underline{2x^4 - 4x^3} \\
-3x^3 + 0x^2 \\
\underline{-3x^3 + 6x^2} \\
-6x^2 + 5x \\
\underline{-6x^2 + 12x} \\
-7x + 4 \\
\underline{-7x + 14} \\
-10
\end{array}
$$

$Q(x) = 2x^3 - 3x^2 - 6x - 7$

$R = -10$

Method 2: Synthetic Division

This example can be solved by only using the coefficients of the terms. Zero coefficients are used for any missing terms.

$$
\begin{array}{r}
2 - 3 - 6 - 7 \\
1 - 2 \overline{)\,2 - 7 + 0 + 5 + 4} \\
\underline{2 - 4} \\
-3 + 0 \\
\underline{-3 + 6} \\
-6 + 5 \\
\underline{-6 + 12} \\
-7 + 4 \\
\underline{-7 + 14} \\
-10
\end{array}
$$

In synthetic division, the coefficient of x in the divisor is always 1. This 1 will be dropped and -2 will be shown instead of $1 - 2$. Notice that the coefficients of the quotient 2, -3, -6, and -7 are repeated in the division procedure.

To avoid unnecessary repetition, each is written only once.

The division process is changed to the following form to avoid these repetitions.

$$
\begin{array}{r}
2-3-6-7 \\
-2\overline{)\,2-7+0+5+4} \\
\underline{-4} \\
-3+0+5+4 \\
\underline{+6+12} \\
-6+5+14 \\
\underline{+12} \\
-7+4 \\
\underline{+14} \\
-10
\end{array}
$$

The process is further simplified by not bringing down the $+0$, $+5$, and $+4$. The remaining numbers in the division process (-4, $+6$, $+12$, and $+14$) and the numbers -3, -6, -7, and -10 are rearranged as follows:

$$
\begin{array}{r}
2-3-6-7 \quad \leftarrow \text{top row} \\
-2\overline{)\,2-7+0+5+4} \\
\underline{-4+6+12+14} \\
-3-6-7-10 \quad \leftarrow \text{bottom row}
\end{array}
$$

If the first coefficient (2) of the top row is brought down to the first position of the bottom row, then the first four numbers of the bottom row are the same as the coefficients of the quotient on the top row.

To further reduce repetition, the top row is deleted and a compact form is obtained as follows:

$$
\begin{array}{r|l}
-2 & 2-7+0+5+4 \leftarrow \text{row 1} \\
\hline
 & -4+6+12+14 \leftarrow \text{row 2} \\
 & 2-3-6-7\boxed{-10} \leftarrow \text{row 3}
\end{array}
$$

The numbers 2, -3, -6, and -7 in row 3 are the coefficients of the quotient $2x^3-3x^2-6x-7$. The remainder is -10.

Check the previous statement to see if it agrees with the long division method. You have probably concluded that the last four numbers in row 3 are obtained by subtracting the numbers in row 2 from row 1.

The following steps explain the synthetic division method for dividing $2x^4-7x^3+5x+4$ by $x-2$.

Step 1: Arrange the polynomial $P(x)$, which is the dividend, in descending powers of x. Any missing terms should be replaced with a zero coefficient.

Step 2: The numbers in row 1 to the right of the vertical line are the coefficients a_n, a_{n-1},..., a_0 of the dividend. Zero is used to represent the coefficient of the missing term $\left(0x^2\right)$. The divisor (-2) to the left of the vertical line represents the second term of the divisor $x-2$.

Step 3: The first coefficient of row 1 is 2. The 2 is written first in row 3; then this 2 is multiplied by the divisor (-2). The product (-4) is written in row 2 under the -7 in row 1.

Step 4: Subtract -4 from -7 and write -3 in row 3.

Step 5: This -3 is multiplied by -2 (the divisor), and the product (6) is placed in row 2 under the 0 in row 1.

Step 6: Subtract 6 from 0, which gives -6. Then, write -6 in row 3.

Step 7: Multiply -6 by -2 (the divisor). The product (12) is placed in row 2 under the 5 from row 1. Subtract 12 from 5, which gives -7.

Step 8: Multiply the -7 by -2 (the divisor). The product (14) is placed in row 2 under the 4 from row 1. Subtract 14 from 4, which gives -10.

Step 9: The last number (-10) in row 3 is the remainder. The remainder has a box around it to keep it separate from the other numbers in row 3.

Step 10: The numbers in row 3, excluding the remainder, represent the coefficients of the quotient. Since the dividend is of the fourth degree and the divisor is of the first degree, the quotient is of the third degree. The quotient is $2x^3 - 3x^2 - 6x - 7$. (Remember: The degree of the quotient polynomial will always be one less than the degree of $P(x)$ since a divisor of degree one is being used.)

Example 2

Divide the polynomial $4x^3 - 3x^2 - 5x - 10$ by $x - 2$ using synthetic division.

Solution

$$\begin{array}{r|rrrr} -2 & 4 & -3 & -5 & -10 \\ & & -8 & -10 & -10 \\ \hline & 4 & 5 & 5 & \boxed{0} \end{array} \leftarrow \text{remainder}$$

The quotient is $4x^2 + 5x + 5$. The remainder is 0.

Example 3

Divide the polynomial $x^4 - 9x^2 - 11$ by $x + 3$ using synthetic division.

Solution

$$x^4 - 9x^2 - 11 = 1x^4 + 0x^3 - 9x^2 + 0x - 11$$

$$\begin{array}{r|rrrrr} 3 & 1 & 0 & -9 & 0 & -11 \\ & & 3 & -9 & 0 & 0 \\ \hline & 1 & -3 & 0 & 0 & \boxed{-11} \end{array} \leftarrow \text{remainder}$$

The quotient is $1x^3 - 3x^2 + 0x + 0$, or $x^3 - 3x^2$. The remainder is -11.

1. Using synthetic division, determine the quotient and the remainder when $x^5 - 2x^3 - 4x^2 - 6x + 12$ is divided by $x - 3$.

2. Use synthetic division to calculate the quotient and the remainder when $-5x^3 + 2x^2 - 3$ is divided by $x + 2$.

Check your answers by turning to the Appendix.

The same principle applies when the division contains a constant term that is a rational number.

Example 4

Use synthetic division to find the quotient and the remainder when $9x^4 - 7x^2 + 14x - 4$ is divided by $x - \frac{1}{3}$.

Solution

$$\begin{array}{r|rrrrr} -\frac{1}{3} & 9 & 0 & -7 & 14 & -4 \\ & & -3 & -1 & 2 & -4 \\ \hline & 9 & 3 & -6 & 12 & 0 \end{array}$$

The quotient is $9x^3 + 3x^2 - 6x + 12$. The remainder is 0.

Since the remainder is zero, then $x - \frac{1}{3}$ is a factor of $9x^4 - 7x^2 + 14x - 4$.

3. Use synthetic division to find the quotient and the remainder when $4x^3 - 6x^2 + 2x + 10$ is divided by $x + \frac{1}{2}$.

Check your answers by turning to the Appendix.

The same principle of divisibility applies to polynomial functions. A polynomial function $P(x)$ is divisible by another polynomial function $D(x)$ if the remainder R is zero.

Example 5

Use synthetic division to find the quotient and the remainder when $P(x) = -2x^4 + 5x^3 + 3x^2 - x + 3$ and $D(x) = x - 3$.

Solution

$$\begin{array}{r|rrrrr} -3 & -2 & 5 & 3 & -1 & 3 \\ & & 6 & 3 & 0 & 3 \\ \hline & -2 & -1 & 0 & -1 & 0 \end{array}$$

$Q(x) = -2x^3 - x^2 - 1$

$R = 0$

Since the remainder is zero, then $x - 3$ is a factor of $-2x^4 + 5x^3 + 3x^2 - x + 3$.

4. Use synthetic division to find the quotient and the remainder when $P(x)=x^4-6x^2-27$ and $D(x)=x+3$.

Check your answers by turning to the Appendix.

Example 6

Use synthetic division to divide $P(x)=6x^3+x^2-1$ by $D(x)=2x-1$. If possible, express $P(x)$ as the product of two integral polynomials.

Solution

Since the synthetic division method needs a coefficient of one for the x-term in the divisor, then change $2x-1$ to $2\left(x-\frac{1}{2}\right)$. Now divide $6x^3+x^2-1$ by $x-\frac{1}{2}$. Remember that the coefficients must be divided by 2 in order to obtain the proper quotient for the divisor $2x-1$.

$$\begin{array}{c|cccc} -\frac{1}{2} & 6 & 1 & 0 & -1 \\ & & -3 & -2 & -1 \\ \hline & 6 & 4 & 2 & 0 \end{array}$$

The quotient coefficients 6, 4, and 2 are for the divisor $x-\frac{1}{2}$.

To obtain the quotient for the divisor $2x-1$, divide these coefficients by 2. Thus, $Q(x)=3x^2+2x+1$ for the divisor $2x-1$. Since the remainder is zero, $\left(x-\frac{1}{2}\right)$ is a factor of $6x^3+x^2-1$.

$$P(x)=\left(x-\tfrac{1}{2}\right)\left(6x^2+4x+2\right)$$

It is possible to express $P(x)$ as the product of two integral polynomials.

$$P(x)=\left(x-\tfrac{1}{2}\right)\left(6x^2+4x+2\right)$$

$P(x)=[D(x)\times Q(x)]+R$

$$=\left(x-\tfrac{1}{2}\right)(2)\left(3x^2+2x+1\right)$$

The common factor of 2 is removed from $Q(x)$.

$$=(2x-1)\left(3x^2+2x+1\right)$$

$\left(x-\frac{1}{2}\right)$ is multiplied by 2.

Thus, $6x^3+x^2-1=(2x-1)\left(3x^2+2x+1\right)$, where $D(x)=2x-1$, $Q(x)=3x^2+2x+1$, and $R=0$.

5. a. Use synthetic division to divide $4x^4+7x^3-14x^2-13x+4$ by $x-\frac{1}{4}$. Then, state the quotient and the remainder.

b. If $x-\frac{1}{4}$ is a factor of the polynomial in 5.a., express that polynomial as a product of two integral polynomials.

6. **a.** Use synthetic division to divide $5x^5 + x^4 - 10x^3 - 2x^2 + 5x + 1$ by $x + \frac{1}{5}$. Then, state the quotient and the remainder.

   b. If $x + \frac{1}{5}$ is a factor of the polynomial in 6.a., express that polynomial as a product of two integral polynomials.

7. **a.** Use synthetic division to divide $3x^4 - x^3 - 6x^2 - 7x + 3$ by $3x - 1$.
   b. State the quotient and the remainder when the divisor is $3x - 1$.
   c. If possible, express the polynomial in 7.a. as the product of two integral polynomials.

Check your answers by turning to the Appendix.

To find the quotient and the remainder when a polynomial is divided by a binomial, the synthetic division method is quicker than the long division method. This method is very helpful in finding all the factors of a polynomial as you will see later in this module.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

In Section 1 you divided polynomial functions using long division and synthetic division. Synthetic division is a useful method with which you will want to be very comfortable.

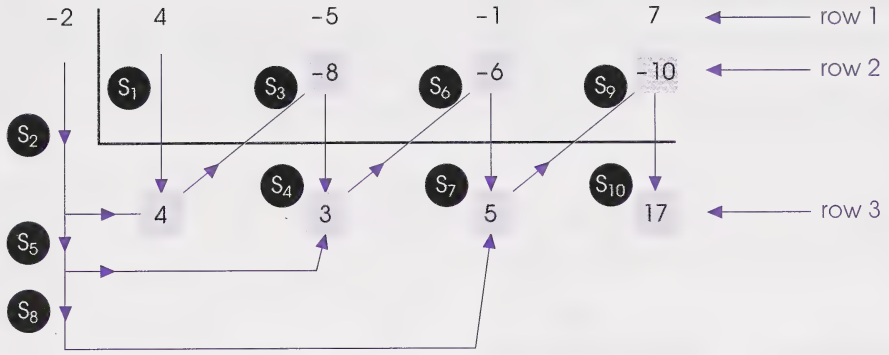
If you had difficulty with synthetic division, a flow chart may help you understand the various steps.

Example

Use synthetic division to find the quotient and the remainder when $4x^3 - 5x^2 - x + 7$ is divided by $x - 2$.

Solution

In this solution, a diagram will be used to summarize the steps in the division procedure. Note that S_1 means Step 1.



Steps:

S_1 Bring down the initial coefficient (4) from row 1 to row 3.

S_2 Multiply -2 and 4. The product is -8.

S_3 Place -8 in row 2 as shown.

S_4 Subtract -8 from -5. The difference is 3.

S_5 Multiply -2 and 3. The product is -6.

S_6 Place -6 in row 2 as shown.

S_7 Subtract -6 from -1. The difference is 5.

S_8 Multiply -2 and 5. The product is -10

S_9 Place -10 in row 2 as shown.

S_{10} Subtract -10 from 7. The difference is 17.

The quotient is $4x^2 + 3x + 5$. The coefficients for the quotient are obtained from row 3, and the degree of the quotient is one less than the degree of $P(x)$. The remainder is 17 (the last term in row 3).

1. Use synthetic division to find the quotient and the remainder when $7x^4 - 2x^3 - 5x - 3$ is divided by $x + 3$.

2. Use synthetic division to find the quotient and the remainder when $2x^3 - 7x^2 + 4x$ is divided by $x - 1$.

3. Use synthetic division to divide $P(x) = 2x^5 - 3x^4 - 2x^3 - 8x - 4$ by $D(x) = 2x + 1$. If possible, express $P(x)$ as the product of two integral polynomials.

4. Use synthetic division to divide $P(x) = -x^4 + 14x^2 - x + 4$ by $D(x) = x + 4$.

Check your answers by turning to the Appendix.

Enrichment

In Activity 4, a subtraction method was used in the synthetic division process. This subtraction method can be modified to a method involving addition. You may prefer the addition method. Most people use addition to avoid errors in sign. It is easier to add than it is to subtract.

Example

Use synthetic division to find the quotient and the remainder when $4x^3 - 5x^2 - x + 7$ is divided by $x - 2$.

Solution

Method 1: Subtraction Method

$$\begin{array}{r|cccc} -2 & 4 & -5 & -1 & 7 \\ & & -8 & -6 & -10 \\ \hline & 4 & 3 & 5 & \boxed{17} \end{array} \leftarrow \text{remainder}$$

The quotient is $4x^2 + 3x + 5$. The remainder is 17.

Method 2: Addition Method

In the subtraction method, the divisor is -2. For the addition method, change -2 to 2. The products obtained by multiplying by 2 are added.

Notice the sign change. $\rightarrow$

$$\begin{array}{r|cccc} 2 & 4 & -5 & -1 & 7 \\ & & 8 & 6 & 10 \\ \hline & 4 & 3 & 5 & \boxed{17} \end{array} \leftarrow \text{remainder}$$

The quotient is $4x^2 + 3x + 5$. The remainder is 17.

Note: When you express the polynomial $P(x)$ in the form $[D(x) \times Q(x)] + R$, make sure you have the correct sign for $D(x)$. The divisor is $x - 2$. It is not $x + 2$.

$$P(x) = [D(x) \times Q(x)] + R$$

$$4x^3 - 5x^2 - x + 7 = (x - 2)(4x^2 + 3x + 5) + 17$$

1. Use synthetic division (addition method) to find the quotient and the remainder when $2x^4 - x^3 + 3x - 8$ is divided by $x - 3$. Express the dividend in the form $[D(x) \times Q(x)] + R$.

2. Use synthetic division (addition method) to find the quotient and the remainder when $8x^3 - 5$ is divided by $x + \frac{1}{2}$. Express the dividend in the form $[D(x) \times Q(x)] + R$.

3. **a.** Use the addition method of synthetic division to divide $2x^4 + 7x^3 - x^2 - 7x + 15$ by $x + 3$.

   b. State the quotient and the remainder.

   c. If possible, express the polynomial in 3.a. as the product of two integral polynomials.

4. If $P(x) = [D(x) \times Q(x)] + R$, state a condition that is necessary for $D(x)$ to be a factor of $P(x)$.

5. If $P(x) = [D(x) \times Q(x)] + R(x)$, where $R(x) \neq 0$, and the degree of $D(x)$ is 2, then explain why the degree of $R(x)$ is 0 or 1.

6. If $P(x)$ is of degree 9, $D(x)$ is of degree 5, and $P(x) = [Q(x) \times D(x)] + R(x)$, then what is the degree of $Q(x)$? What can the highest degree of $R(x)$ be?

Check your answers by turning to the Appendix.

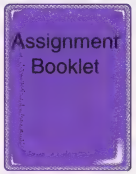
Conclusion

In this section you identified and classified polynomial functions. You also discovered how to divide polynomials using the synthetic division method, which has an advantage over the long division method. The synthetic division method shortens the process and can be used to reduce a polynomial to its factored form.

Polynomial functions may be used to model a variety of everyday situations. The formula for determining the height of a bridge from the time it takes a stone to hit the water below is just one example. In the sections that follow, you will encounter additional applications, which will require a deeper understanding of the polynomial function.

Gaining an understanding of polynomials will help you build a repertoire of skills for solving real-world problems.

Assignment



You are now ready to complete the section assignment.

Section 2: Remainder Theorem and Factor Theorem

What happens to a golf ball when a golfer tees off? The upward motion of the ball begins slowing due to gravity until it reaches a maximum distance above the ground. At this moment, it begins to accelerate on its return to the ground. If the vertical height d (in metres) of the ball after a certain time t (in seconds) is given by the polynomial function $d = -5t^2 + 40t$, what is the ball's position after five seconds?

In this section you will use direct substitution as one method of answering this and other similar questions. You will discover another method of evaluating a polynomial function, called the Remainder Theorem. The Remainder Theorem evaluates a polynomial using division rather than using direct substitution.

In addition to the Remainder Theorem, you will study a related theorem, called the Factor Theorem, for analysing polynomial functions. You will use the Factor Theorem to decide whether or not a particular binomial is a factor of a given polynomial function.

Activity 1: Evaluating Polynomial Functions

Recall the golf ball problem in the section introduction. If height d (in metres) and time t (in seconds) is given by the polynomial function $d = -5t^2 + 40t$, what is the position of the ball after it is hit?

If $t = 2$ s,

$$\begin{aligned} d &= -5t^2 + 40t \\ &= -5(2)^2 + 40(2) \\ &= -20 + 80 \\ &= 60 \end{aligned}$$

The ball reaches a height of 60 m.

If $t = 5$ s, $d = -5t^2 + 40t$

$$\begin{aligned} &= -5(5)^2 + 40(5) \\ &= -125 + 200 \\ &= 75 \end{aligned}$$

The ball reaches a height of 75 m.

When evaluating a polynomial function, a given value is substituted for the variable, and a calculated value is obtained.

The polynomial function is usually shown by the symbol $P(x)$ (read ***P* of *x***). If the variable x in the polynomial function is replaced by a real number, the polynomial function represents a real number. When a real number m is substituted for x in $P(x)$, the real number used is called the value of $P(x)$ at m and is represented by $P(m)$.

Example 1

If $P(x) = 2x^3 - x^2 + 3x - 4$, calculate $P(-2)$.

$P(-2)$ is read as ***P* at −2**.

Solution

$$P(x) = 2x^3 - x^2 + 3x - 4$$

$$\begin{aligned} P(-2) &= 2(-2)^3 - (-2)^2 + 3(-2) - 4 \\ &= 2(-8) - (4) - 6 - 4 \\ &= -16 - 4 - 6 - 4 \\ &= -30 \end{aligned}$$

Example 2

If $P(x)=2x^3-x^2+3x-4$, calculate $P\left(\sqrt{2}\right)$.

Solution

$$P(x)=2x^3-x^2+3x-4$$
$$P\left(\sqrt{2}\right)=2\left(\sqrt{2}\right)^3-\left(\sqrt{2}\right)^2+3\left(\sqrt{2}\right)-4$$
$$=2\left(2\sqrt{2}\right)-2+3\sqrt{2}-4$$
$$=4\sqrt{2}-2+3\sqrt{2}-4$$
$$=7\sqrt{2}-6$$

Example 3

If $P(x)=2x^3-x^2+3x-4$, calculate $P(d)$.

Solution

$$P(x)=2x^3-x^2+3x-4$$
$$P(d)=2(d)^3-d^2+3(d)-4$$
$$=2d^3-d^2+3d-4$$

1. If $P(x)=-x^4+2x^2+3x-8$, calculate the following:

a. $P(-1)$ **b.** $P(2)$ **c.** $P\left(\sqrt{3}\right)$

d. $P(0)$ **e.** $P(k)$

2. For the polynomial function $P(x)=3x^2-5$, calculate the following:

a. $P\left(\sqrt{5}\right)$ **b.** $P\left(\frac{1}{3}\right)$ **c.** $P(0)$

d. $P(-2)$ **e.** $P(k)$

3. Given $P(x)=2x^3-3x^2-8$, calculate the following:

a. $P\left(-\frac{1}{2}\right)$ **b.** $P(-2)$ **c.** $P(3)$

d. $P\left(\frac{1}{3}\right)$ **e.** $P(0)$ **f.** $P(k)$

Check your answers by turning to the Appendix.

Did you realize that values substituted for x are elements of the domain of the function? The calculated values of $P(x)$ are elements of the range of the function.

Activity 2: The Remainder Theorem

In Section 1 you divided polynomial functions. You discovered that sometimes there is a remainder. In this activity, you will discover an interesting fact about the remainder.

1. Given the polynomial functions and the divisor for each polynomial function, determine the remainder R for each polynomial function by using synthetic division. Then, complete a chart like the one given. Leave the right column blank.

$P(x)$	Divisor	R	
$-3x^2-4x+5$	$x-3$		
$4x^2+x-8$	$x-3$		
$-2x^3-3x+7$	$x-3$		
$3x^3-2x^2-3$	$x-3$		

2. Calculate $P(3)$ for each polynomial function in question 1. Then add the heading $P(3)$ to the blank column and complete the chart.

3. Examine the chart you completed in question 2. Compare the values in the last two columns. What pattern do you notice?

Check your answers by turning to the Appendix.

Did you notice that the remainder values are the same as the values of $P(3)$? This is always the case. Look at the following table.

$P(x)$	Divisor	R	$P(2)$
x^2-2x-5	$x-2$	-5	-5
$2x^2-x+3$	$x-2$	9	9
x^3-2x^2-4	$x-2$	-4	-4
$2x^4-x^3-2x-1$	$x-2$	19	19

Notice that the remainder values are the same as the values of $P(2)$. In general, if a polynomial function is divided by $x-a$, then the remainder is equal to the value of the polynomial when $x=a$.

If a polynomial $P(x)$ is divided by $x-a$, the remainder R is equal to $P(a)$. This is called the **Remainder Theorem**.

The division algorithm $P(x)=[D(x)\times Q(x)]+R$ was developed in Section 1. Using $x-a$ for the divisor, the division algorithm can be written in the form $P(x)=[(x-a)\times Q(x)]+R$.

$$\therefore \text{ if } x=a,\ P(a)=[(a-a)\times Q(x)]+R$$
$$=[0\times Q(x)]+R$$
$$=R$$

This proves the Remainder Theorem.

The following examples show how the Remainder Theorem can be used to find a remainder.

Example 1

Find the remainder when the polynomial function $P(x)=2x^2-5x+8$ is divided by $x-1$.

Solution

Method 1: Synthetic Division

$$\begin{array}{r|rrr} -1 & 2 & -5 & 8 \\ & & -2 & 3 \\ \hline & 2 & -3 & \boxed{5} \end{array} \leftarrow \text{remainder}$$

The remainder is 5.

Method 2: Remainder Theorem

If the polynomial function $P(x)=2x^2-5x+8$ is divided by $x-1$, the remainder R is equal to $P(1)$.

$$P(x)=2x^2-5x+8$$
$$P(1)=2(1)^2-5(1)+8$$
$$=2-5+8$$
$$=5$$
$$R=P(1)$$
$$=5$$

The remainder is 5.

Example 2

Calculate the remainder when the polynomial function $P(x)=3x^3-4x+7$ is divided by $x+3$.

Solution

Method 1: Synthetic Division

$$\begin{array}{r|rrrr} 3 & 3 & 0 & -4 & 7 \\ & & 9 & -27 & 69 \\ \hline & 3 & -9 & 23 & \boxed{-62} \end{array} \leftarrow \text{remainder}$$

The remainder is -62.

Method 2: Remainder Theorem

If the polynomial function $P(x)=3x^3-4x+7$ is divided by $x+3$, the remainder is equal to $P(-3)$.

$$\begin{aligned} P(x)&=3x^3-4x+7\\ P(-3)&=3(-3)^3-4(-3)+7\\ &=-81+12+7\\ &=-62\end{aligned}$$

$$\begin{aligned} R&=P(-3)\\ &=-62\end{aligned}$$

The remainder is -62.

4. Use the Remainder Theorem to calculate the remainder for each of the following. Use synthetic division to check the value of the remainder.

a. Divide $P(x)=-2x^3+6x-4$ by $x-2$.

b. Divide $P(x)=2x^3-x-3$ by $x+1$.

c. Divide $P(x)=-x^4+2x^3-x+4$ by $x+2$.

Check your answers by turning to the Appendix.

The following examples show how the Remainder Theorem can be used to evaluate a polynomial function.

Example 3

Evaluate $P(x)=2x^3-8x^2+3x+5$ when $x=-4$.

Solution

Method 1: Remainder Theorem

$$\begin{aligned} P(x)&=2x^3-8x^2+3x+5\\ P(-4)&=2(-4)^3-8(-4)^2+3(-4)+5\\ &=-128-128-12+5\\ &=-263\end{aligned}$$

Method 2: Synthetic Division

If the polynomial function $P(x)=2x^3-8x^2+3x+5$ is divided by $x+4$, the remainder R is equal to $P(-4)$.

4	2	−8	3	5
		8	−64	268
	2	−16	67	−263 ← remainder

Since $R=-263$, then $P(-4)=-263$.

Example 4

Evaluate $P(x)=x^7-4x^4+x-3$ when $x=2$.

Solution

Method 1: Remainder Theorem

$$P(x)=x^7-4x^4+x-3$$
$$P(2)=(2)^7-4(2)^4+2-3$$
$$=128-64+2-3$$
$$=63$$

Method 2: Synthetic Division

If the polynomial function $P(x)=x^7-4x^4+x-3$ is divided by $x-2$, the remainder R is equal to $P(2)$.

-2	1	0	0	-4	0	0	1	-3
		-2	-4	-8	-8	-16	-32	-66
	1	2	4	4	8	16	33	63 ← remainder

Since $R=63$, then $P(2)=63$.

5. Use the Remainder Theorem to do the following.

a. Evaluate $P(x)=x^5-3x^4+3x-4$ when $x=2$.

b. Evaluate $P(x)=3x^3-9x^2-3x+4$ when $x=-3$.

Check your answers by turning to the Appendix.

The following example shows how the Remainder Theorem can be used to determine the value of a numerical coefficient.

Example 5

When $P(x)=2x^4-x^3-dx+8$ is divided by $x-3$, the remainder is 131. Determine the value of d.

Solution

If the polynomial function $P(x)=2x^4-x^3-dx+8$ is divided by $x-3$, the remainder is equal to $P(3)$.

$$P(x)=2x^4-x^3-dx+8$$
$$P(3)=2(3)^4-(3)^3-d(3)+8$$
$$131=162-27-3d+8$$
$$131=143-3d$$
$$3d=12$$
$$d=4$$

6. a. When $P(x)=3x^3-2x^2-dx-4$ is divided by $x+2$, the remainder is -34. Determine the value of d.

b. When $P(x)=-4x^4+2x^3+dx^2-3x+9$ is divided by $x+1$, the remainder is 4. Determine the value of d.

Check your answers by turning to the Appendix.

The Remainder Theorem can also be used to determine the values of two numerical coefficients.

Example 6

If $P(x)=-2x^2+dx+c$, $P(1)=-9$, and $P(-3)=-13$, determine the values of d and c.

Solution

$$P(1)=-2(1)^2+d(1)+c$$
$$-9=-2+d+c$$
$$d+c=-7 \quad (1)$$

$$P(-3)=-2(-3)^2+d(-3)+c$$
$$-13=-18-3d+c$$
$$-3d+c=5 \quad (2)$$

Subtract (2) from (1).

$$\begin{aligned} d+c&=-7 \quad (1)\\ -3d+c&=5 \quad (2)\\ \hline 4d&=-12\\ d&=-3 \end{aligned}$$

Substitute $d=-3$ into (1).

$$d+c=-7$$
$$-3+c=-7$$
$$c=-4$$

7. When $P(x)=2x^4+dx^3-cx^2+5x-8$ is divided by $x+3$, it gives a remainder of -41. When $P(x)$ is divided by $x-2$, it gives a remainder of 74. Determine the values of d and c.

Check your answers by turning to the Appendix.

Did you find that using the Remainder Theorem is quicker than using synthetic division when finding the remainder?

Activity 3: The Factor Theorem

In this module you have been examining the remainders when a polynomial function has been divided by a binomial.

You have discovered that the binomial is a factor of the polynomial function if the remainder is zero.

The division algorithm $P(x)=[D(x)\times Q(x)]+R$ was developed in Section 1. Using $x-a$ for the divisor, the division algorithm can be written in the form $P(x)=[(x-a)\times Q(x)]+R$.

If $R=0$, then $P(x)=(x-a)\times Q(x)$, where $x-a$ and $Q(x)$ are factors.

A polynomial function $P(x)$ has the binomial $x-a$ as a factor if, and only if, $P(a)=0$. This is called the **Factor Theorem**.

The following examples show how the Factor Theorem can be used to test for factors of a polynomial function.

Example 1

Is $x-4$ a factor of the polynomial function $P(x)=x^3-5x^2+3x+4$?

Solution

Since $x-a$ is a factor if $P(a)=0$, then $x-4$ is a factor only if $P(4)=0$.

$$\begin{aligned}P(4)&=4^3-5(4)^2+3(4)+4\\&=64-80+12+4\\&=0\end{aligned}$$

Since $P(4)=0$, then $x-4$ is a factor of $P(x)=x^3-5x^2+3x+4$.

Example 2

Is $x+3$ a factor of the polynomial $P(x)=-2x^3-6x^2+3x-9$?

Solution

Since $x-a$ is a factor if $P(a)=0$, then $x+3$ is a factor if $P(-3)=0$.

$$\begin{aligned}P(-3)&=-2(-3)^3-6(-3)^2+3(-3)-9\\&=54-54-9-9\\&=-18\end{aligned}$$

Since $P(-3)\neq 0$, then $x+3$ is **not** a factor.

Use the Factor Theorem to determine if the binomial $D(x)$ is a factor of the polynomial $P(x)$.

1. $P(x)=2x^4-6x^3+8x^2-2x-2$
$D(x)=x-1$

2. $P(x)=3x^3-2x^2+5x+2$
$D(x)=x+\frac{1}{3}$

3. $P(x)=-2x^4+x^3+7x^2-3x+6$
$D(x)=x+2$

4. $P(x)=2x^4-7x^3-4x^2+6x+3$
$D(x)=x+\frac{1}{2}$

5. $P(x)=4x^3+3x^2-8x-6$
$D(x)=x+\sqrt{2}$

Check your answers by turning to the Appendix.

You can use the Factor Theorem to factor an integral polynomial function completely. You will use this theorem in the next section to determine the zeros of the polynomial functions.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

In this section you were introduced to the Remainder Theorem and the Factor Theorem. Use the information presented in this section to answer the following questions.

1. Fill in the blanks.

a. When evaluating a polynomial function, a given value is substituted for the _____, a calculated value is obtained for $P(x)$.

b. If a polynomial $P(x)$ is divided by $x-a$, the _____ is equal to $P(a)$.

c. A polynomial $P(x)$ has the binomial $x-a$ as a _____ only if $P(a)=0$.

d. The number a is a _____ of a polynomial only if $P(a)=0$.

2. Find the remainder for each if $P(x) = -3x^4 - 2x^2 - 8$.

a. $P(-2)$ **b.** $P(\sqrt{2})$

3. Use the Remainder Theorem to find the remainder when each polynomial is divided by the given binomial. Use synthetic division to check the remainder.

a. $P(x) = 3x^4 - 4x^3 + 5x - 8$
$D(x) = x - 2$

b. $P(x) = x^3 - 4x + 9$
$D(x) = x + 4$

4. When $P(x) = -x^4 - 5x^3 + 4x^2 - dx + 3$ is divided by $x + 2$, the remainder is 61. Calculate the value of d.

Check your answers by turning to the Appendix.

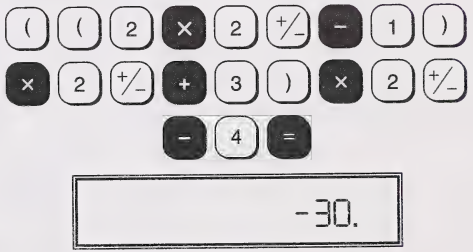
Enrichment

Example 1 in Activity 1 of this section can be done in less time by using the **nested form** and a calculator.

The nested form is a collection of sets where the common variable is factored and the terms are grouped. In order to get the nested form of $P(x)$, do the following:

$$\begin{aligned} P(x) &= 2x^3 - x^2 + 3x - 4 \\ &= \left[2x^2 - x + 3\right]x - 4 \\ &= \left[(2x - 1)x + 3\right]x - 4 \end{aligned}$$

Evaluate $P(-2)$ using a calculator.



Note: The calculator you are using may require a different method for finding the answer.

It is more difficult to evaluate roots with a calculator. Direct substitution of the root in the polynomial is the better method.

Use the nested form and a calculator to evaluate $4x^3 - 5x^2 - 2x + 4$ for the given value of x.

1. $f(3)$ **2.** $f(-9)$ **3.** $f(6)$

Check your answers by turning to the Appendix.

Conclusion

In this section you used direct substitution to evaluate polynomial functions. You also obtained a relationship between a polynomial and a binomial divisor. This relationship, described by the Remainder Theorem, may be used to connect the remainders obtained by division by binomials with the values of polynomials obtained by substitution. An extension of the Remainder Theorem, called the Factor Theorem, was used to decide whether or not a given binomial was a factor of a particular polynomial.

The trajectory of a golf ball, from the time it leaves the head of a golf club to the point at which the ball lands, can be represented by a polynomial function. The Remainder Theorem and the Factor Theorem form the basis for describing the trajectory of the golf ball. These theorems are used to describe the position of the golf ball at any given time, or vice versa. In later sections you will use the Remainder Theorem and the Factor Theorem to analyse polynomial functions in more detail.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Section 3: Solving Polynomial Equations

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In the complex environment of a greenhouse, a gardener may have to consider a variety of issues separately – from light and moisture to temperature and disease control. In a like manner, a mathematician, when working with polynomial functions, may have to break the function down into parts dealing with simpler and more familiar forms. You might say they are both getting to the "root" of the problem.

Mathematicians have spent a lot of time and effort to solve polynomial equations. It was not until the sixteenth century that formulas were discovered for finding the zeros of third- and fourth-degree polynomials. Modern mathematics has shown that fifth-degree and higher polynomial functions do not have general solutions.

In this section you will determine integral zeros and rational zeros of polynomial functions. You will master techniques to factorize integral polynomial functions and solve polynomial equations.

Activity 1: The Integral Zero Theorem

In this module you have been examining polynomial functions and polynomial equations.

For example, $P(x)=2x^2+6x+4$ is a polynomial function, and $2x^2+6x+4=0$ is a polynomial equation.

As you can see, the polynomial equation $x^2+3x+2=0$ is made up of a polynomial.

Do not confuse a polynomial function with a polynomial equation.

You can divide a polynomial equation by a number and maintain its equality.

$$2x^2+6x+4=0$$
$$x^2+3x+2=0$$

You cannot divide a polynomial function by a number and maintain its equality.

$$f(x)=2x^2+6x+4$$
$$g(x)=x^2+3x+2$$
$$f(x)\neq g(x)$$

1. State whether each of the following is a polynomial function or a polynomial equation. Explain.

a. $y=x^2-3x+2$ **b.** $3x^2-5x-1=0$

c. $y=2x^2-8x+4$ **d.** $3x^2-6x=9$

Check your answers by turning to the Appendix.

If x is replaced with -1, the polynomial x^2+3x+2 will equal 0.

If $x=-1$, $x^2+3x+2=(-1)^2+3(-1)+2$

$=1-3+2$

$=0$

Since replacing x with -1 makes the polynomial equal to zero, then -1 is a zero of the polynomial.

A **zero of a polynomial** is a value of x which makes the polynomial zero in value. If $P(a)=0$; then a is a zero of $P(x)$.

The relation $y=x^2+3x+2$ is a polynomial function. If x is replaced with -1 in the polynomial function $y=x^2+3x+2$, then y equals 0.

If $x=-1$, $y=x^2+3x+2$

$$=(-1)^2+3(-1)+2$$
$$=1-3+2$$
$$=0$$

Since replacing x with -1 makes the polynomial function equal to 0, then -1 is the zero of the polynomial function.

A **zero of a polynomial function** is a value for the polynomial function which makes the polynomial function zero.

As you see, the polynomial function $y=x^2+3x+2$ is an equation. The ordered pair $(-1,0)$ or $x=-1$ and $y=0$ makes $y=x^2+3x+2$ a true equality.

You can verify this by substituting $x=-1$ and $y=0$ into the function.

LS	RS
y	x^2+3x+2
$=0$	$=(-1)^2+3(-1)+2$
	$=1-3+2$
	$=0$

LS = RS

Since $x=-1$ and $y=0$ makes the polynomial function a true equality, then -1 is a **root** of the polynomial function.

A root of an equation makes the equation true.

Do not confuse the terms *zero* and *root*. **Polynomials** have **zeros**, but they do not have roots. **Polynomial functions** have **zeros**. Zeros of a polynomial function are roots of the polynomial equation obtained when $y=0$.

When y is a different number, the polynomial function has different roots which are not zeros.

2. a. Is -3 a zero of the polynomial x^2+5x+6? Why or why not?

b. Is -3 a root of the polynomial x^2+5x+6? Why or why not?

3. a. Is -3 a zero of the polynomial function $P(x)=x^2-9$? Why or why not?

b. Is -3 a root of the polynomial function $P(x)=x^2-9$? Why or why not?

Check your answers by turning to the Appendix.

In Section 1 you classified polynomials as integral, rational, and real according to their coefficients.

Zeros can be classified as integral, rational, and real as well.

If the zero is an integer, it is an **integral zero**. If the zero is a rational number, it is a **rational zero**. If the zero is an irrational number, it is a **real zero**.

In this activity you will develop a method to find the integral zeros of an integral polynomial function.

Example 1

What are the integral zeros of $P(x)=3x-2$?

Solution

If v is an integral zero of $P(x)$, then $P(v)=3v-2$ and $P(v)=0$.

$$\begin{aligned} \therefore\ 3v-2 &= 0 \\ 3v &= 2 \\ v &= \frac{2}{3} \end{aligned}$$

Since $\frac{2}{3}$ is not an integer, $P(x)$ does not have any integral zeros.

Note: Not all integral polynomials have integral zeros.

Example 2

What are the integral zeros of $P(x)=x^2-1$?

Solution

If v is an integral zero of $P(x)$, then $P(v)=v^2-1$ and $P(v)=0$.

$$\begin{aligned} \therefore\ v^2-1 &= 0 \\ v^2 &= 1 \\ v &= \pm 1 \end{aligned}$$

The integral zeros of $P(x)=x^2-1$ are 1 and -1.

Example 3

What are the zeros of $P(x)=2x^3+7x^2-17x-10$?

Solution

$$\begin{aligned} P(x) &= 2x^3+7x^2-17x-10 \\ &= x\left(2x^2+7x-17\right)-10 \end{aligned}$$

If v is an integral zero of $P(x)$, then

$P(v)=v\left(2v^2+7v-17\right)-10$ and $P(v)=0$.

$$\therefore\ v\left(2v^2+7v-17\right)-10=0$$
$$v\left(2v^2+7v-17\right)=10$$

Since v is an integer, then $2v^2+7v-17$ must also be an integer because the set of integers is closed under multiplication, addition, and subtraction. It is now evident that v is a factor of 10. Thus, the only integral values for v which could possibly be zeros of $P(x)$ are the integral factors of 10, which are ±1, ±2, ±5, and ±10.

Thus, these integral factors of 10 are the **potential integral zeros** of $P(x)$. Now, these potential integral zeros can be checked to see which, if any, are integral zeros of the polynomial. Synthetic division or direct substitution of the potential integral zeros can be used for this checking procedure.

Is 1 a zero?

$$P(x)=2x^3+7x^2-17x-10$$
$$P(1)=2(1)^3+7(1)^2-17(1)-10$$
$$=2+7-17-10$$
$$=-18$$

Since $P(1)\neq0$, then 1 is not a zero.

Is -1 a zero?

$$P(x)=2x^3+7x^2-17x-10$$
$$P(-1)=2(-1)^3+7(-1)^2-17(-1)-10$$
$$=-2+7-17-10$$
$$=12$$

Since $P(-1)\neq0$, then -1 is not a zero.

Is 2 a zero?

$$\begin{array}{r|cccc} -2 & 2 & 7 & -17 & -10 \\ & & -4 & -22 & -10 \\ \hline & 2 & 11 & 5 & \boxed{0} \end{array} \leftarrow \text{remainder}$$

Since the remainder is zero, $x-2$ is a factor and $P(2)=0$.

Recall that the Factor Theorem states that $x-a$ is a factor of $P(x)$ if and only if $P(a)=0$.

The original polynomial function can now be expressed in the form $P(x)=(x-2)\left(2x^2+11x+5\right)$.

Is -2 a zero?

Any other factors of $P(x)$ must be factors of the quotient $Q(x)=2x^2+11x+5$.

The constant term in $Q(x)$ is 5; thus, -2, 10, and -10 are not integral factors of this constant term. Remember that 1 and -1 were checked previously, but were not integral zeros. This leaves 5 and -5 to be tested.

Are 5 or -5 zeros? When continuing this check using synthetic division, use the coefficients of $Q(x)$. Do not go back to the original $P(x)$.

$$\begin{array}{r|rrr} -5 & 2 & 11 & 5 \\ & & -10 & -105 \\ \hline & 2 & 21 & \boxed{110} \end{array}$$

Since the remainder is not zero, $x-5$ is not a factor and 5 is not a zero.

$$\begin{array}{r|rrr} 5 & 2 & 11 & 5 \\ & & 10 & 5 \\ \hline & 2 & 1 & \boxed{0} \end{array}$$

Since the remainder is zero, $x+5$ is a factor and -5 is a zero.

Therefore, $P(x)=2x^3+7x^2-17x-10$ has two integral zeros: 2 and -5. This polynomial has one more zero but it is nonintegral. This zero comes from the quotient $2x+1$. If $2x+1=0$, $x=-\frac{1}{2}$.

You have observed that an integral zero is a factor of the constant term of the integral polynomial function.

This leads you to a statement regarding the Integral Zero Theorem.

If an integer is a zero of an integral polynomial function $P(x)$, then this same integer is a factor of the constant term a_0 of $P(x)$. This is called the **Integral Zero Theorem.**

Now look at some examples involving the Integral Zero Theorem.

Example 4

List all potential integral zeros of $P(x)=3x^4+8x^3-7x^2+6x-20$.

Solution

The potential integral zeros are factors of the constant -20. Thus, the potential integral zeros are ±1, ±2, ±4, ±5, ±10, and ±20.

4. List all the potential integral zeros for each polynomial function.

a. $P(x)=4x^3-3x^2+5x-7$

b. $P(x)=3x^4-x^3+2x-8$

c. $P(x)=-6x^4+2x^2+5x-9$

d. $P(x)=8x^2-3x+10$

e. $P(x)=-6x^3+5x-12$

Check your answers by turning to the Appendix.

Example 5

Determine all the integral zeros of $P(x)=3x^3+11x^2-19x+5$.

Solution

Method 1

The potential integral zeros are factors of the constant term 5. Thus, the potential integral zeros are ± 1 and ± 5.

These values can be substituted into $P(x)$ to see which ones produce a remainder of zero.

$$\begin{aligned} P(1) &= 3(1)^3+11(1)^2-19(1)+5 \\ &= 3+11-19+5 \\ &= 0 \end{aligned}$$

Thus, 1 is a zero.

$$\begin{aligned} P(-1) &= 3(-1)^3+11(-1)^2-19(-1)+5 \\ &= -3+11+19+5 \\ &= 32 \end{aligned}$$

Thus, -1 is not a zero.

$$\begin{aligned} P(5) &= 3(5)^3+11(5)^2-19(5)+5 \\ &= 375+275-95+5 \\ &= 560 \end{aligned}$$

Thus, 5 is not a zero.

$$\begin{aligned} P(-5) &= 3(-5)^3+11(-5)^2-19(-5)+5 \\ &= -375+275+95+5 \\ &= 0 \end{aligned}$$

Thus, -5 is a zero.

Therefore, 1 and -5 are the two integral zeros. The third zero of this polynomial function is not an integer.

Method 2

The potential integral zeros 1, -1, 5, and -5 may be checked using synthetic division rather than using substitutions of the potential zeros.

Let the divisor be 1.

1	3	11	−19	5
		3	8	−27
	3	8	−27	32

Since the remainder is not zero, $x+1$ is not a factor and $x=-1$ is not a zero.

Now try -1 as the divisor.

$$\begin{array}{r|rrrr} -1 & 3 & 11 & -19 & 5 \\ & & -3 & -14 & 5 \\ \hline & 3 & 14 & -5 & \boxed{0} \end{array}$$

Since the remainder is zero, $x-1$ is a factor and $x=1$ is a zero.

The polynomial function $P(x)$ can now be expressed as $P(x)=(x-1)\left(3x^2+14x-5\right)$. The remaining zeros may be determined by factoring the quotient $Q(x)=3x^2+14x-5$.

Let $Q(x)=0$.

$$\therefore\ 3x^2+14x-5=0$$
$$(3x-1)(x+5)=0$$
$$(3x-1)=0 \quad \text{or} \quad x+5=0$$
$$3x=1 \qquad\qquad x=-5$$
$$x=\frac{1}{3}$$

The two integral zeros are 1 and -5. The third zero is $\frac{1}{3}$. $\frac{1}{3}$ is not an integral zero, but it is a rational zero.

Example 6

Use synthetic division to determine all integral zeros of $P(x)=x^4+x^3-4x^2-2x+4$.

Solution

The potential integral zeros are ±1, ±2, and ±4 (factors of the constant 4).

$$\begin{array}{r|rrrrr} 1 & 1 & 1 & -4 & -2 & 4 \\ & & 1 & 0 & -4 & 2 \\ \hline & 1 & 0 & -4 & 2 & \boxed{2} \end{array}$$

Since the remainder is not zero, $x+1$ is not a factor and $x=-1$ is not a zero.

$$\begin{array}{r|rrrrr} -1 & 1 & 1 & -4 & -2 & 4 \\ & & -1 & -2 & 2 & 4 \\ \hline & 1 & 2 & -2 & -4 & \boxed{0} \end{array}$$

Since the remainder is zero, $x-1$ is a factor and $x=1$ is a zero.

If $x-a$ is the divisor and the remainder is zero, then $x=a$ is a zero of the polynomial.

Thus, $P(x)=(x-1)\left(1x^3+2x^2-2x-4\right)$. The remaining zeros of $P(x)$ are zeros of $Q(x)=1x^3+2x^2-2x-4$. Use synthetic division to find the remaining zeros. The potential zero -1 does not need to be checked since it was not a zero in the first part of the method. Check the zero 1 again since it may be repeated.

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -2 & -4 \\ & & -1 & -3 & -1 \\ \hline & 1 & 3 & 1 & \boxed{-3} \end{array}$$

Since the remainder is not zero, the zero 1 does not repeat.

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -2 & -4 \\ & & 2 & 0 & -4 \\ \hline & 1 & 0 & -2 & \boxed{0} \end{array}$$

Since the remainder is zero, $x+2$ is a factor and -2 is an integral zero. Thus, $P(x)=(x-1)(x+2)\left(x^2-2\right)$.

The remaining zeros of $P(x)$, in addition to 1 and -2, are also zeros of x^2-2. These zeros can be determined by solving the quadratic equation $x^2-2=0$.

$$x^2-2=0$$
$$\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)=0$$
$$x-\sqrt{2}=0 \quad \text{or} \quad x+\sqrt{2}=0$$
$$x=\sqrt{2} \qquad x=-\sqrt{2}$$

Both final zeros are irrational. Thus, the integral zeros are 1 and -2.

If the quadratic equation cannot be factored, then use the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

5. Determine the integral zeros for each polynomial.

a. $P(x)=2x^3+3x^2-18x+8$

b. $P(x)=x^3-7x^2+16x-12$

c. $P(x)=4x^3+15x^2-132x+32$

d. $P(x)=x^4+7x^3+6x^2-28x-40$

Check your answers by turning to the Appendix.

You should have a clear picture between zeros and roots for polynomial functions. You should also have a good understanding of classifying zeros (as integral, rational, or real zeros).

Activity 2: The Rational Zero Theorem

In Activity 1 you determined the integral zeros of an integral polynomial function. You also noticed that an integral polynomial function may contain rational zeros.

In this activity you will develop a method to find the rational zeros of a polynomial function.

A **rational number** has the form $\frac{a}{b}$, where a and b are relatively prime integers and $b \neq 0$.

> Two integers are relatively prime if they have no common factors other than $+1$ or -1.

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$ and $\frac{v}{w}$ is a zero of $P(x)$, then $a_n \left(\frac{v}{w}\right)^n + a_{n-1} \left(\frac{v}{w}\right)^{n-1} + \cdots + a_1 \left(\frac{v}{w}\right) + a_0 = 0$.

Multiplying by w^n gives

$$a_n v^n + a_{n-1} v^{n-1} w + \cdots + a_1 v w^{n-1} + a_0 w^n = 0.$$

Now $a_0 w^n$ is subtracted from both sides and v is factored from the left side. This gives

$$v\left(a_n v^{n-1} + a_{n-1} v^{n-2} w + \cdots + a_1 w^{n-1}\right) = -a_0 w^n.$$

Integers are closed under the operations of addition, subtraction, and multiplication. This means that if two integers are added, the sum is another integer. If two integers are multiplied, the product is also an integer. Thus, the quantity inside the parentheses must be an integer. The left side is now the product of two integers; that is, v times an integer $= -a_0 w^n$. Thus, v is a factor of $-a_0 w^n$. Since v has no factor in common with w, it has no factor in common with w^n; therefore, v is a factor of a_0.

The equation $a_n v^n + a_{n-1} v^{n-1} w + \cdots + a_1 v w^{n-1} + a_0 w^n = 0$ can be changed to $a_{n-1} v^{n-1} w + \cdots + a_1 v w^{n-1} + a_0 w^n = -a_n v^n$.

Now, w is factored from the left side.

$$w\left(a_{n-1} v^{n-1} + \cdots + a_1 v w^{n-2} + a_0 w^{n-1}\right) = -a_n v^n$$

Since w has no factor in common with v, it has no factor in common with v^n; thus, w is a factor of a_n.

From this theory, the following theorem for finding rational roots is now summarized.

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$ is a polynomial function with integral coefficients and $\frac{v}{w}$ is a rational number in lowest terms which is a zero of $P(x)$, then v is a factor of a_0 and w is a factor of a_n. This is called the **Rational Zero Theorem**.

For $\frac{v}{w}$ to be in lowest terms and to be a zero of $P(x)$, v must be a factor of the constant a_0 and w must be a factor of the coefficient of the highest power (a_n).

In other words, potential numerators are factors of the constant term (last term), and potential denominators are factors of the coefficient of the highest degree term (first term). The potential rational zeros are all possible combinations of these factors when the potential numerators are divided by the potential denominators.

What are the factors of the constant term of the polynomial function $P(x)=-2x^3-5x^2+25x+3$? The constant term a_0 is 3. Therefore, the factors of 3 are ±1 and ±3. Keep in mind that these factors are the potential numerators of the rational number $\frac{v}{w}$ and will represent v.

The numerical coefficient of the highest degree term is -2 and $a_n=-2$.

What are the factors of -2?

The factors of -2 are ±1 and ±2. These factors are the potential denominators of the rational number $\frac{v}{w}$ and will represent w. These potential numerators and potential denominators are combined to form the potential rational zeros.

Begin by dividing ±1, which are factors of 3, by all the factors of -2. The potential rational zeros are $\pm\frac{1}{1}$ and $\pm\frac{1}{2}$. Then divide ±3, which are factors of 3, by all the factors of -2. The potential rational zeros are $\pm\frac{3}{1}$ and $\pm\frac{3}{2}$.

When you rearrange the zeros, you will get the potential rational zeros ±1, ±3, $\pm\frac{1}{2}$, and $\pm\frac{3}{2}$.

Before studying the following example, remember that potential integral zeros are contained in the set of potential rational zeros. When checking the potential rational zeros, always check the integral values first. If you had to find the actual zeros, you would start by checking the integers ±1 and ±3.

The following example shows the method for finding the zeros of the polynomial function.

Example

Determine all the rational zeros of $P(x)=5x^3-51x^2+55x-9$.

Solution

The potential numerators are ±1, ±3, and ±9.
The potential denominators are ±1 and ±5.
The potential rational zeros are ±1, ±3, ±9, $\pm\frac{1}{5}$, $\pm\frac{3}{5}$, and $\pm\frac{9}{5}$.

Always start by finding any integral zeros.

1	5	−51	55	−9
		5	−56	111
	5	−56	111	−120

Since the remainder is not zero, $x+1$ is not a factor and -1 is not a zero.

$$\begin{array}{r|rrrr} -1 & 5 & -51 & 55 & -9 \\ & & -5 & 46 & -9 \\ \hline & 5 & -46 & 9 & 0 \end{array}$$

Since the remainder is zero, $x-1$ is a factor and 1 is a zero.

The polynomial function can be written as
$P(x)=(x-1)\left(5x^2-46x+9\right)$.

The remaining zeros are obtained from $Q(x)=\left(5x^2-46x+9\right)$.

If possible, factor $\left(5x^2-46x+9\right)$. If it is not possible to factor the quadratic, then use the quadratic formula.

$$\left(5x^2-46x+9\right)=(5x-1)(x-9)$$

To obtain the remaining zeros, let $(5x-1)(x-9)=0$.

$$\begin{array}{lll} 5x-1=0 & \text{or} & x-9=0 \\ 5x=1 & & x=9 \\ x=\frac{1}{5} & & \end{array}$$

The rational zeros are 1, 9, and $\frac{1}{5}$.

1. List all the potential rational zeros of each polynomial.

a. $P(x)=2x^3+4x^2-3x+8$

b. $P(x)=3x^4+2x^3-x^2+5x-9$

c. $P(x)=-5x^3-2x+6$

d. $P(x)=4x^3+6x^2-8x-10$

e. $P(x)=8x^3-4x^2+5x-15$

2. Determine the rational zeros for each polynomial.

a. $P(x)=8x^3-10x^2+x+1$

b. $P(x)=18x^3-15x^2-x+2$

c. $P(x)=8x^4-6x^3-19x^2-3x+2$

d. $P(x)=4x^3+7x^2+2x-1$

Check your answers by turning to the Appendix.

In this activity you recognized that all rational zeros of a polynomial function are of the form $\frac{v}{w}$, where v is a factor of a_0 and w is a factor of a_n.

Activity 3: Factoring Integral Polynomial Functions

If you can find the zeros of a polynomial, then you can factor it.

The steps in factoring an integral polynomial over the integers are as follows:

Step 1: Determine the potential rational zeros. The potential numerators are factors of the constant term a_0, and the potential denominators are factors of the initial coefficient a_n.

Step 2: Substitute values of the potential rational zeros directly into the polynomial. When substituting, use the integral values first. If a value results in the polynomial being equal to zero, then that value is a zero of the polynomial. If a is a zero of the polynomial, then $x-a$ is a factor. For example, if 5 is a zero, then $x-5$ is a factor. If -3 is a zero, then $x-(-3)$, which equals $x+3$, is a factor. This direct substitution could be used to check all the potential zeros, but it could result in a time-consuming process, especially if there are many potential zeros.

Step 3: Synthetic division can be used from the start. If the remainder is zero and the divisor is $-a$, then the factor is $x-a$. For example, if the divisor is -2, then $x-2$ is a factor and 2 is a zero. If the divisor is a when the remainder is 0, then $x+a$ is the factor. For example, if 3 is the divisor, then $x+3$ is the factor and -3 is the zero. After one zero is obtained, the polynomial is written as the product of the binomial factor and the quotient polynomial obtained using synthetic division. The degree of the quotient polynomial is one less than the degree of the polynomial. If this quotient polynomial is of degree 2, then it can be factored (if possible) to find the remaining factors and the zeros. If the quotient polynomial cannot be factored, then it can be solved using the quadratic formula. If the quotient polynomial is of degree 3 and higher, then continue using synthetic division. This method is now applied to the quotient polynomial, not to the original polynomial. Since a number may be a multiple zero, then check the quotient polynomial to see if the root repeats. Also, if a potential zero has been tried and found not to be a zero of the original polynomial, it will not be a zero of the quotient polynomial at any point; thus, it does not have to be tried again.

Sometimes it is better to use direct substitution to obtain one zero, then switch to synthetic divison to obtain the other zeros.

Step 4: When you factor an integral polynomial over the integers, all the factors must have integral coefficients and constants. For example, the polynomial $P(x)=6x^4-14x^3-5x^2+21x-6$ can be expressed in the factored form as $P(x)=(x-2)\left(x-\frac{1}{3}\right)\left(6x^2-9\right)$.

This factored form comes from the rational zeros found. The polynomial is not in the factored form over the integers because of the $-\frac{1}{3}$ (which is not an integer). The factored form $(x-2)\left(x-\frac{1}{3}\right)\left(6x^2-9\right)$ is over the rationals.

Notice how this rational form must be changed to obtain the proper integral form.

$$\begin{aligned} P(x) &= (x-2)\left(x-\frac{1}{3}\right)\left(6x^2-9\right) \\ &= (x-2)\left(x-\frac{1}{3}\right)(3)\left(2x^2-3\right) \\ &= (x-2)(3x-1)\left(2x^2-3\right) \end{aligned}$$

The common factor 3 is removed.

Thus,

$P(x)=(x-2)(3x-1)\left(2x^2-3\right)$

is the factored form of $P(x)$ over the integers. The real zeros of $P(x)$ are 2, $\frac{1}{3}$, $\frac{+\sqrt{6}}{2}$, and $\frac{-\sqrt{6}}{2}$.

$$\begin{aligned} 2x^2-3 &= 0 \\ x^2 &= \frac{3}{2} \\ x &= \pm\sqrt{\frac{3}{2}} \\ &= \pm\frac{\sqrt{6}}{2} \end{aligned}$$

Study the following example. If you are having any problems, refer to the steps given at the beginning of this activity.

Example

Factor the integral polynomial $P(x)=3x^4-4x^3-31x^2-24x-4$ over the integers, and determine all of its real zeros.

Solution

The potential numerators are ±1, ±2, and ±4 (factors of the constant term).

The potential denominators are ±1 and ±3 (factors of the initial coefficient).

The potential rational zeros are ±1, ±2, ±4, $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, and $\pm\frac{4}{3}$.

Use synthetic division to solve the problem after direct substitution is used to find one zero.

$$\begin{aligned} P(1) &= 3(1)^4-4(1)^3-31(1)^2-24(1)-4 \\ &= 3-4-31-24-4 \\ &= -60 \end{aligned}$$

Thus, 1 is not a zero.

$$P(-1)=3(-1)^4-4(-1)^3-31(-1)^2-24(-1)-4$$
$$=3+4-31+24-4$$
$$=-4$$

Thus, -1 is not a zero.

$$P(2)=3(2)^4-4(2)^3-31(2)^2-24(2)-4$$
$$=48-32-124-48-4$$
$$=-160$$

Thus, 2 is not a zero.

$$P(-2)=3(-2)^4-4(-2)^3-31(-2)^2-24(-2)-4$$
$$=48+32-124+48-4$$
$$=0$$

Thus, -2 is a zero.

Since -2 is a zero, then $x+2$ is a factor. Start with $+2$ in the divisor of the synthetic division procedure.

$+2$	3	-4	-31	-24	-4
		6	-20	-22	-4
	3	-10	-11	-2	0

Since the remainder is zero, then $x+2$ is a factor and the zero is -2. This checks with the direct substitution.

$$P(x)=(x+2)\left(3x^3-10x^2-11x-2\right)$$

$$P(x)=D(x)\times Q(x)$$

Now use synthetic division on the quotient polynomial $3x^3-10x^2-11x-2$. Check the zero -2 in case the -2 repeats in the quotient polynomial.

$+2$	3	-10	-11	-2
		6	-32	42
	3	-16	21	-44

The remaining zeros in addition to -2 are found from the quotient polynomial.

Since the remainder is not zero, $x+2$ is not a factor and -2 is not a zero of the quotient polynomial.

-4	3	-10	-11	-2
		-12	-8	12
	3	2	-3	-14

Thus, 4 is not a zero.

4	3	-10	-11	-2
		12	-88	308
	3	-22	77	-310

Thus, -4 is not a zero.

$$\begin{array}{r|cccc} -\frac{1}{3} & 3 & -10 & -11 & -2 \\ & & -1 & 3 & 4\frac{2}{3} \\ \hline & 3 & -9 & -14 & \boxed{-6\frac{2}{3}} \end{array}$$

Thus, $\frac{1}{3}$ is not a zero.

$$\begin{array}{r|cccc} \frac{1}{3} & 3 & -10 & -11 & -2 \\ & & 1 & -\frac{11}{3} & -\frac{22}{9} \\ \hline & 3 & -11 & -7\frac{1}{3} & \boxed{\frac{4}{9}} \end{array}$$

Thus, $-\frac{1}{3}$ is not a zero.

$$\begin{array}{r|cccc} -\frac{2}{3} & 3 & -10 & -11 & -2 \\ & & -2 & \frac{16}{3} & \frac{98}{9} \\ \hline & 3 & -8 & -16\frac{1}{3} & \boxed{-12\frac{8}{9}} \end{array}$$

Thus, $\frac{2}{3}$ is not a zero.

$$\begin{array}{r|cccc} \frac{2}{3} & 3 & -10 & -11 & -2 \\ & & 2 & -8 & -2 \\ \hline & 3 & -12 & -3 & \boxed{0} \end{array}$$

Since the remainder is zero, $x+\frac{2}{3}$ is a factor and $x=-\frac{2}{3}$ is a zero.

Therefore, $P(x)=(x+2)\left(x+\frac{2}{3}\right)\left(3x^2-12x-3\right)$. The polynomial function $P(x)$ is not in the factored form over the integers because of the $+\frac{2}{3}$ (which is a rational number).

The factors must be changed in order to obtain the correct form.

$$\begin{aligned} P(x) &= (x+2)\left(x+\frac{2}{3}\right)(3)\left(x^2-4x-1\right) \\ &= (x+2)(3x+2)\left(x^2-4x-1\right) \end{aligned}$$

Any further factors of $P(x)$ must be factors of x^2-4x-1.

Note that x^2-4x-1 cannot be factored any further.

Since the only possible rational zeros of x^2-4x-1 are ± 1, and neither of these work, then $P(x)=(x+2)(3x+2)\left(x^2-4x-1\right)$ is the factored form of $P(x)$ over the integers.

Since the quadratic is not factorable over the integers, the zeros for $x^2-4x-1=0$ must be determined by the quadratic formula.

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad (\text{where } a=1,\ b=-4,\ \text{and } c=-1) \\ &= \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-1)}}{2(1)} \\ &= \frac{4\pm\sqrt{20}}{2} \\ &= \frac{4\pm 2\sqrt{5}}{2} \\ &= 2\pm\sqrt{5} \end{aligned}$$

Some important statements will now be summarized.

$$P(x)=3x^4-4x^3-31x^2-24x-4$$

The factored form of $P(x)$ over the integers is

$$P(x)=(x+2)(3x+2)\left(x^2-4x-1\right).$$

The real zeros of $P(x)$ are -2, $-\frac{2}{3}$, $2+\sqrt{5}$, and $2-\sqrt{5}$.

$P(x)$ can be factored into first-degree factors as shown, but this form is not required.

$$P(x)=(x+2)(3x+2)\left(x-2-\sqrt{5}\right)\left(x-2+\sqrt{5}\right)$$

This is not a factored form over the integers since $\sqrt{5}$ is not an integer.

The expression x^2-4x-1 cannot be factored further to obtain factors with integral coefficients and integral constants.

Factor the integral polynomial functions over the integers, and determine all the real zeros.

1. $P(x)=2x^3-3x^2-11x+6$

2. $P(x)=x^4-3x^3-2x^2+12x-8$

3. $P(x)=6x^3+7x^2-1$

Check your answers by turning to the Appendix.

There are several techniques that can be used to factor polynomial functions. You should be comfortable to find the zeros or factors of a polynomial function.

Activity 4: Solving Polynomial Equations

So far you have been able to solve only linear and quadratic equations. You will now be able to solve a polynomial equation of higher degree.

A **polynomial equation** is a polynomial expression equated to zero.

Polynomial equations of degree greater than two are difficult to solve. If the polynomial can be factored, then the roots can be obtained. Solving for the roots of a polynomial equation is similar to finding the zeros of a polynomial expression.

Example 1

Solve $x^4-3x^3-3x^2+16x-12=0$.

Solution

Solving this equation for its roots is similar to finding the zeros of $P(x)=x^4-3x^3-3x^2+16x-12$.

The potential numerators are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$, and ± 12 (factors of the constant).

The potential denominators are ± 1 (factors of the initial coefficient).

The potential rational zeros are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$, and ± 12.

The potential rational zeros are the potential numerators divided by the potential denominators.

$$\begin{aligned} P(1) &= (1)^4 - 3(1)^3 - 3(1)^2 + 16(1) - 12 \\ &= 1 - 3 - 3 + 16 - 12 \\ &= -1 \end{aligned}$$

Thus, 1 is not a zero.

$$\begin{aligned} P(-1) &= (-1)^4 - 3(-1)^3 - 3(-1)^2 + 16(-1) - 12 \\ &= 1 + 3 - 3 - 16 - 12 \\ &= -27 \end{aligned}$$

Thus, -1 is not a zero.

$$\begin{aligned} P(2) &= (2)^4 - 3(2)^3 - 3(2)^2 + 16(2) - 12 \\ &= 16 - 24 - 12 + 32 - 12 \\ &= 0 \end{aligned}$$

Thus, 2 is a zero.

Since 2 is a root, $x-2$ is a factor. Use -2 as the divisor in the synthetic division method.

$$\begin{array}{r|rrrrr} -2 & 1 & -3 & -3 & 16 & -12 \\ & & -2 & 2 & 10 & -12 \\ \hline & 1 & -1 & -5 & 6 & \boxed{0} \end{array}$$

Since the remainder is zero, $x-2$ is a factor and 2 is a root. This checks with direct substitution.

At this stage, the factored form of the polynomial is $(x-2)\left(x^3 - x^2 - 5x + 6\right)$. The remaining roots are roots of the quotient polynomial $x^3 - x^2 - 5x + 6$. The first root (2) should be checked a second time in the quotient polynomial in case the root repeats. To check for the root 2, you have to start with a divisor of -2.

$$\begin{array}{r|rrrr} -2 & 1 & -1 & -5 & 6 \\ & & -2 & -2 & 6 \\ \hline & 1 & 1 & -3 & \boxed{0} \end{array}$$

Since the remainder is zero, $x-2$ is a factor and 2 is a root. Note that the root 2 does repeat. Since the root 2 occurs twice, it is said to have a **multiplicity** of two.

Multiplicity of a root is the number of times the same root occurs in the solution.

Now the factored form is $(x-2)(x-2)\left(x^2+x-3\right)$. The remaining roots are roots of the quotient polynomial x^2+x-3. The quadratic formula must be used since this polynomial cannot be factored by ordinary methods.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad (\text{where } a=1,\ b=1,\ \text{and } c=-3)$$
$$=\frac{-(1)\pm\sqrt{(1)^2-4(1)(-3)}}{2(1)}$$
$$=\frac{-1\pm\sqrt{13}}{2}$$

All the roots of the equation $x^4-3x^3-3x^2+16x-12=0$ are 2 (multiplicity of two), $\frac{-1+\sqrt{13}}{2}$, and $\frac{-1-\sqrt{13}}{2}$.

Example 2

Solve $20x^3-17x^2-5x+2=0$.

Solution

Solving this equation for its roots is similar to determining the zeros of $P(x)=20x^3-17x^2-5x+2$.

The potential numerators are ±1 and ±2 (factors of the constant term).

The potential denominators are ±1, ±2, ±4, ±5, ±10, and ±20 (factors of the initial coefficient).

The potential rational zeros are ±1, ±2, $\pm\frac{1}{4}$, $\pm\frac{1}{2}$, $\pm\frac{1}{5}$, $\pm\frac{2}{5}$, $\pm\frac{1}{10}$, and $\pm\frac{1}{20}$.

$$P(1)=20(1)^3-17(1)^2-5(1)+2$$
$$=20-17-5+2$$
$$=0$$

Thus, 1 is a zero and $x-1$ is a factor. Start with -1 in the divisor of the synthetic division method.

$$\begin{array}{r|rrrr} -1 & 20 & -17 & -5 & 2 \\ & & -20 & -3 & 2 \\ \hline & 20 & 3 & -2 & \boxed{0} \end{array}$$

Since the remainder is zero, then $x-1$ is a factor and 1 is a zero. This checks with direct substitution.

Now the factored form is $(x-1)\left(20x^2+3x-2\right)$. The remaining zeros are zeros of the quotient polynomial $20x^2+3x-2$. The quotient polynomial is not a cubic expression. The quadratic may be factored directly, and the zeros determined. If the quadratic cannot be factored directly, use the quadratic formula to calculate the zeros.

$$20x^2 + 3x - 2 = 0$$
$$(4x-1)(5x+2) = 0$$

$4x-1=0$ or $5x+2=0$

$4x=1$ $\qquad$ $5x=-2$

$x=\frac{1}{4}$ $\qquad$ $x=-\frac{2}{5}$

All the roots of $20x^3 - 17x^2 - 5x + 2 = 0$ are 1, $\frac{1}{4}$, and $-\frac{2}{5}$.

Make sure you understand the examples before doing the practice exercises.

1. Solve the following polynomial equations.

a. $6x^3 + 7x^2 - 9x + 2 = 0$
b. $8x^4 - 6x^3 - 53x^2 - 9x + 18 = 0$
c. $x^4 - 3x^3 - 7x^2 + 15x + 18 = 0$
d. $2x^4 + 9x^3 + 4x^2 - 16x - 8 = 0$

Check your answers by turning to the Appendix.

Look at an example which is a real-life situation.

Example 3

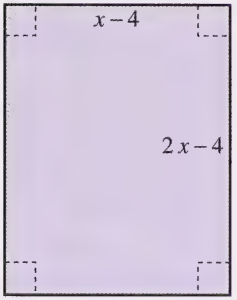
Two-centimetre squares are cut from each corner of a rectangular piece of cardboard, which is twice as long as it is wide. The flaps are turned up to form an open box. The volume of the box is 320 cm^3. What are the dimensions of the original piece of cardboard?

Solution

Let x be the width of cardboard.
Let $2x$ be the length of cardboard.

The squares in each corner will reduce each side by 4 cm.

$\therefore\ w = x - 4$ and $l = 2x - 4$



$$v = l \times w \times h$$
$$320 = (2x-4)(x-4)(2)$$
$$320 = 4x^2 - 24x + 32$$
$$0 = 4x^2 - 24x - 288$$
$$0 = x^2 - 6x - 72$$
$$(x-12)(x+6) = 0$$

$x-12=0$ or $x+6=0$

$x=12$ $\qquad$ $x=-6$ (impossible)

$\therefore l = 2x$

$= 2(12)$

$= 24$

The dimensions of the original cardboard are 12 cm by 24 cm.

2. The height of a freezer compartment is 2 m less than the length, and the width is 4 m less than the length. If the freezer holds a volume of $48\ \text{m}^3$, what is the length of the freezer compartment?

3. A rectangular piece of sheet metal measures 16 cm by 20 cm. Squares are cut from each corner of this sheet and the flaps are turned up to form an open box. If the volume of the box is $384\ \text{cm}^3$, what will the side of a square measure?

4. A can in the form of a right circular cylinder has a circular base and top. If the total area of the surface is $301.44\ \text{cm}^2$, and the height is 8 cm, find the volume. (Use $\pi = 3.14$).

Check your answers by turning to the Appendix.

In this activity you discovered how to solve equations. This helped you further to solve real-world problems.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

The following exercises will help you get a clearer understanding of some of the terminology.

Fill in the blanks in questions 1 to 7, and answer questions 8 to 14.

1. An integral polynomial only has coefficients that are _____.

2. The Factor Theorem states that $x - a$ is a factor of $P(x)$ if _____.

3. If an integer is a zero of an integral polynomial, then this same integer is a _____ of the _____ term of $P(x)$.

4. In synthetic division, a zero of the polynomial is determined when the remainder is zero. If the divisor is -3 when the remainder is 0, then _____ is a factor and _____ is a zero. If the divisor is 4 when the remainder is 0, then _____ is a factor and _____ is a zero.

5. From the Rational Zero Theorem, the potential numerators are factors of the _____ term _____. The potential denominators are factors of the coefficient _____ of the _____ degree term.

6. A polynomial _____ is a polynomial expression equated to zero.

7. The _____ of a root is the number of times the same root occurs in the solution.

8. List all potential integral zeros of $P(x)=2x^4-6x^3-3x^2+5x-18$.

9. Determine all integral zeros of $P(x)=2x^3-3x^2-11x+6$.

10. Given the polynomial $P(x)=3x^3-7x^2+5x-7$, determine the potential rational zeros.

11. Determine all the rational zeros of $P(x)=4x^3+17x^2-16x-5$.

12. Factor the integral polynomial $P(x)=2x^4-9x^3+11x^2-x-3$ over the integers, and determine all of its real zeros.

13. Solve $12x^3+17x^2+2x-3=0$.

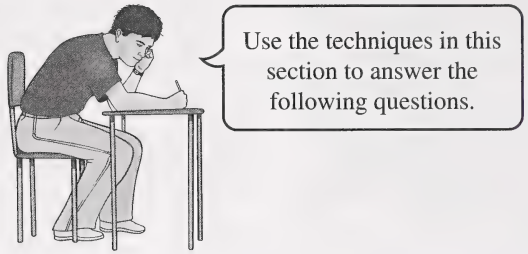
14. Determine the integral polynomial of degree 4 that has the zeros 1, -2, and $\frac{1}{4}$ (multiplicity of two). The polynomial has a remainder of 100 when divided by $x+1$.

Check your answers by turning to the Appendix.

Enrichment

Did You Know?

About 2000 B.C., the Babylonians were able to solve quadratic equations and could find the zeros of some cubic and quartic equations. But it was not until the sixteenth century that formulas that provided the zeros of cubic and quartic equations were determined. In 1824, the Norwegian mathematician, Abel, discovered that it was impossible to obtain formulas that will determine the zeros of quintic equations and polynomial equations of higher degrees.



1. Solve the quintic function $P(x)=x^5-8x^3+6x^2+7x-6$.

2. **a.** Determine the smallest positive value of k if the potential integral zeros of $x^4+4x^3-9x^2-16x+k$ are ±1, ±2, ±4, and ±5.

   b. Using the value of k determined in question 2.a., find the actual integral zeros of $x^4+4x^3-9x^2-16x+k$.

3. Solve the polynomial equation $6x^4-2x^2=13x^3-7x-2$.

4. Explain why no negative number can be a zero of $P(x)=3x^5+2x^3+8x-10$.

5. Just by glancing at $R(x)=2x^6+x^4+3x^3+5$, explain how it is possible to tell that no positive number is a zero of $R(x)$.

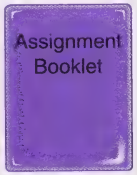
Check your answers by turning to the Appendix.

Conclusion

In this section you gained an understanding on how to develop the Integral Zero Theorem and the Rational Zero Theorem to factor a given integral polynomial over the integers. You also applied the Factor Theorem to solve equations involving polynomials of degree greater than 2 and determined an integral polynomial function given its rational zeros.

Like a gardener controlling the environment so plants can thrive, solving polynomial equations and factoring polynomial expressions also involve controlling factors. In this case, your gardening tools are the Integral Zero Theorem and the Rational Zero Theorem.

Assignment



You are now ready to complete the section assignment.

Section 4: Graphing Polynomial Functions

NASA

Can you imagine what it would be like to be in a spacecraft looking back at Earth? Astronauts have commented on how amazing this experience is. This photograph was taken in 1972 when Apollo 17 was between the Sun and Earth enabling the astronauts to get a full-disk view of Earth.

It has been said that a picture is worth a thousand words. People always like to see videos, photographs, and drawings to help them understand things better.

In this section you will see what polynomial functions look like by examining their graphs. You will discover that such graphs will have particular characteristics depending on the function. You will also determine a polynomial function given certain data or a graph.

Activity 1: Using Technology to Graph a Polynomial Function

In Section 1 of this module, you studied that polynomial functions can be classified by degree. Here are some examples.

- $P(x) = -3x + 6$ is a first degree or linear polynomial function.
- $P(x) = x^2 - 2x - 3$ is a second degree or quadratic polynomial function.
- $P(x) = x^3 - x^2 - 4x + 4$ is a third degree or cubic polynomial function.

Linear functions, as you know, are easy to graph.

Example 1

Graph $y = -3x + 6$.

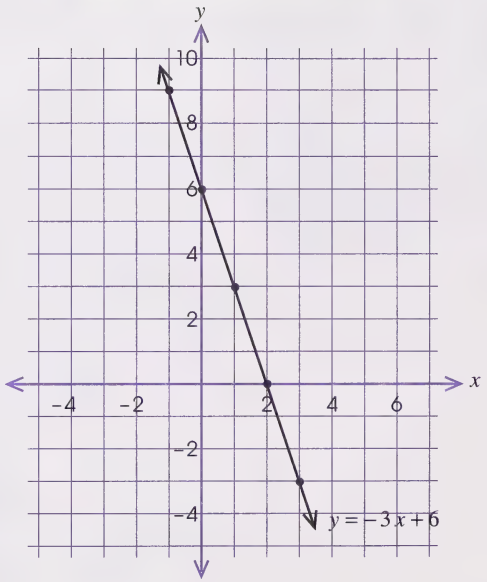
When plotting a graph, $P(x)$ is usually replaced by y.

Solution

Method 1: Using a Table of Values

Make a table of values; then plot the points.

x	-1	0	1	2	3
y	9	6	3	0	-3



Method 2: Determining the Intercepts

Determine the y-intercept and the x-intercept.

The y-intercept occurs when $x = 0$.

$$y = -3(0) + 6$$
$$= 6$$

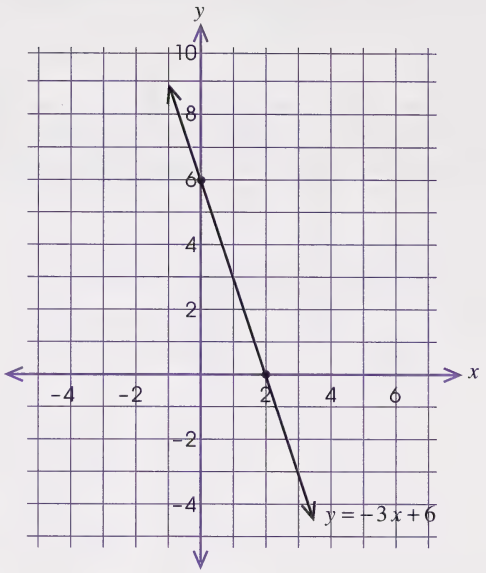
The y-intercept is 6.

The x-intercept occurs when $y = 0$.

$$0 = -3x + 6$$
$$3x = 6$$
$$x = 2$$

The x-intercept is 2.

Plot both intercepts and then join them with a straight line.



You are also familiar with graphing quadratic functions.

Example 2

Graph $y = 2x^2 - 4x - 6$.

Solution

Determine the y-intercept and the x-intercepts.

Let $x = 0$.

$$y = 2(0)^2 - 4(0) - 6$$
$$y = -6$$

Thus, the y-intercept is -6.

Let $y = 0$.

$$0 = 2x^2 - 4x - 6$$
$$0 = 2\left(x^2 - 2x - 3\right)$$
$$0 = 2(x+1)(x-3)$$

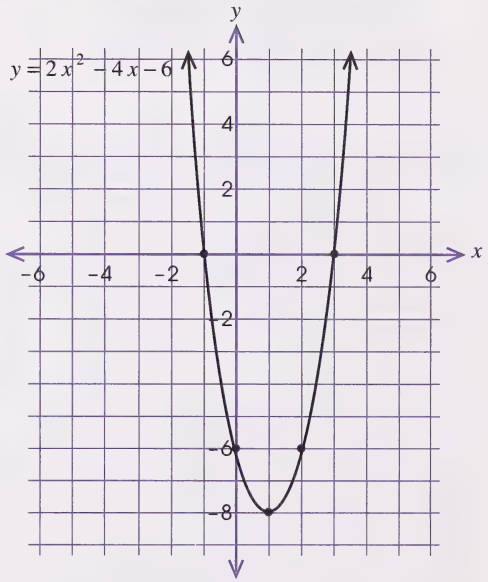
$x + 1 = 0$ or $x - 3 = 0$

$x = -1$ $\qquad$ $x = 3$

Thus, the x-intercepts are -1 and 3.

Prepare a table of values using some x-values close to the zeros. These values provide for a more accurate graph.

x	-1	0	1	2	3
y	0	-6	-8	-6	0



Polynomial functions of degree 3 or higher are more difficult to graph than linear or quadratic functions. When using paper-and-pencil techniques, a simple sketch of the graph is often made.

Example 3

Sketch the graph of $P(x)=x^3-x^2-4x+4$.

Solution

Step 1: To determine the y-intercept, let $x=0$. Thus, the y-intercept is 4.

Step 2: Factor the polynomial; then determine the real zeros from the factors. The potential rational zeros are ±1, ±2, and ±4.

$P(1)=(1)^3-(1)^2-4(1)+4=0$. Thus, 1 is a zero and $x-1$ is a factor. Use -1 as the divisor in the synthetic division method.

$$\begin{array}{r|cccc} -1 & 1 & -1 & -4 & 4 \\ & & -1 & 0 & 4 \\ \hline & 1 & 0 & -4 & \boxed{0} \end{array}$$

Since the remainder is 0, $x-1$ is a factor and 1 is a zero. This checks with direct substitution.

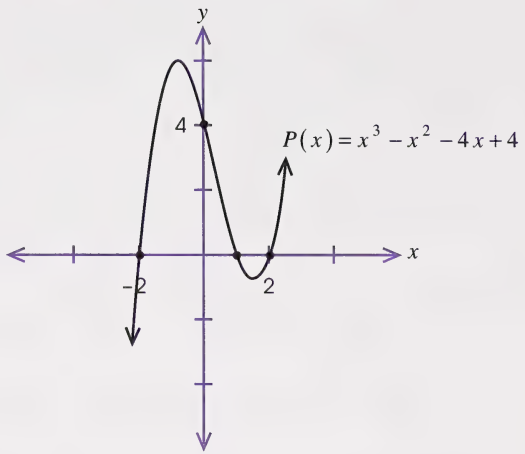
$$\begin{aligned} P(x) &= (x-1)\left(x^2-4\right) \\ &= (x-1)(x+2)(x-2) \end{aligned}$$

Let $P(x)=0$.

$$0=(x-1)(x+2)(x-2)$$

Thus, the x-intercepts are 1, -2, and 2.

Step 3: Plot the y-intercept and the zeros. Then join the points using a smooth curve.



Remember that this is a sketch which shows the y-intercept, the zeros, and the general shape of the polynomial function.

Using a Graphing Calculator

You will now use a graphing calculator to generate graphs of some polynomial functions. Directions are given for one type of graphing calculator – the CASIO *fx-7700G*. If you are using a different graphing calculator, then consult the owner's manual.

To draw the graph given in Example 2 using a CASIO *fx-7700G* graphing calculator, follow these steps:

Step 1: Turn the graphing calculator on. Press [Shift], [F5], and [EXE] to clear the graphing screen.

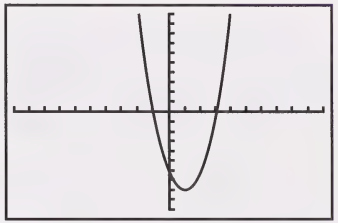
Step 2: Set the range. The following settings are commonly used.

```
Range
Xmin : −10
  max : 10
  scl : 1
Ymin : −10
  max : 10
  scl : 1
INIT
```

Step 3: Press the following keys to graph $P(x)=2x^2-4x-6$.

[Graph] [2] [X,θ,T] [x^y] [2] [−]
[4] [X,θ,T] [−] [6] [EXE]

The graph will appear and look as follows:



Use a graphing calculator to do the following questions.

1. Draw the graph of $y=x^3-3x^2-x+3$. Then determine the y-intercept and the x-intercepts.

2. Draw each graph. Then determine the y-intercept and the x-intercepts.

a. $y=-x^2-2x+3$ **b.** $P(x)=-2x^3-9x^2-7x+6$

To graph a polynomial function where the initial coefficient of the highest degree variable is negative, press the $(-)$ key.

Check your answers by turning to the Appendix.

Graph sketching is made simpler and quicker with the help of technology.

Activity 2: Graphing Polynomial Functions

Graphs of polynomial functions have particular characteristics depending on the function.

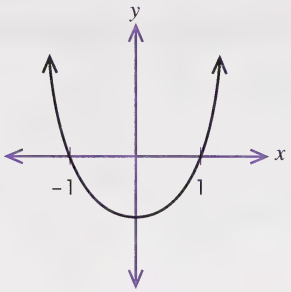
Graphs of polynomial functions are affected by the degree of the polynomial function, the zeros of the polynomial function, and the coefficient of the term with the highest degree.

Degree of the Polynomial

The degree of the polynomial affects the shape of the graph. Look at the first two examples showing the graphs for quadratic and cubic polynomials.

The **degree** of a polynomial is **at least one more** than the number of changes of direction in the graph.

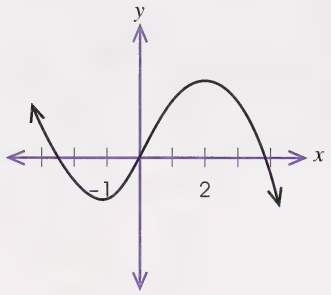
In Graph A, the graph changes direction once at $x = 0$. If you look at the graph from left to right, the graph has a downward direction until $x = 0$, then the direction turns upwards. This is a sketch of a degree 2 (quadratic) polynomial.



Graph A

One direction change plus one equals two, which is the degree of the quadratic polynomial.

In Graph B, the graph changes direction twice (at $x = -1$ and $x = 2$). The sketch represents a degree 3 (cubic) polynomial.



Graph B

Two direction changes plus one equals three, which is the degree of the cubic polynomial.

Zeros of the Polynomial

The zeros of a polynomial may be real or imaginary. **Real zeros** occur at the points where the polynomial crosses or touches the x-axis. If the graph of a polynomial function does not cross or touch the x-axis, then all the **zeros** are **imaginary**.

Real roots include integers and rational and irrational numbers.

Polynomials may have zeros that occur more than once.

The **multiplicity** of a zero is the number of times that the same zero occurs for a polynomial.

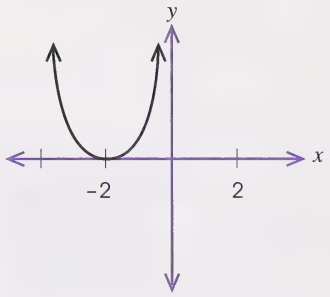
For example, in $P(x)=(x-3)(x+1)^2$, 3 is a zero of multiplicity one and -1 is a zero of multiplicity two.

The **sum** of all the multiplicities of all zeros of a polynomial is equal to the **degree** of the polynomial.

If the multiplicity is **even**, then the graph **touches** the ***x*-axis** at that point. If the multiplicity is **odd**, the graph **crosses** the ***x*-axis** at that point.

Even means 2, 4, 6, …
Odd means 1, 3, 5, 7, …

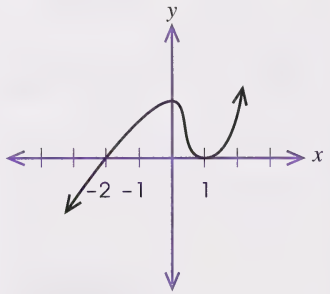
In Graph C, the graph of the polynomial touches the x-axis at -2. The point where the graph touches represents two real and equal roots.



Graph C

The graph is a quadratic polynomial with two real equal zeros. The two zeros are both equal to -2. The multiplicity of -2 is two (i.e., even), and the graph touches at -2.

In Graph D, the graph crosses the x-axis at -2 and touches the x-axis at 1. This polynomial has one zero equal to -2 and two zeros equal to 1. This sketch represents a cubic polynomial.



Graph D

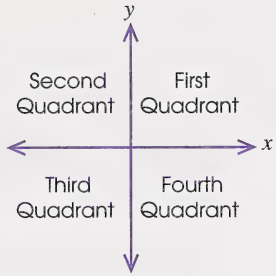
The zero -2 has a multiplicity of one, whereas the zero 1 has a multiplicity of two.

Initial Coefficient of the Highest Degree Variable

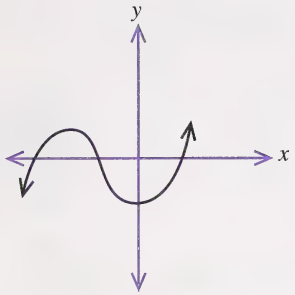
The notation for an integral polynomial function is

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$.

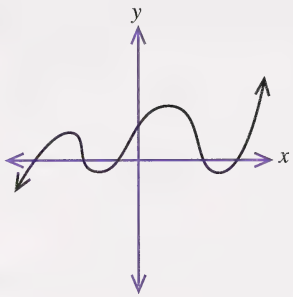
- If $n = 3$ (or any odd integer) and a_n is positive, then, by looking at the graph from left to right, the graph begins by rising in the third quadrant and ends by rising in the first quadrant.



In Graph E, the degree of the polynomial is 3, and a_n is positive. In Graph F, the degree of the polynomial is 5, and a_n is positive.



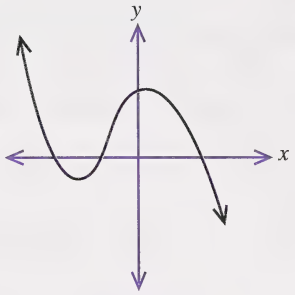
Graph E



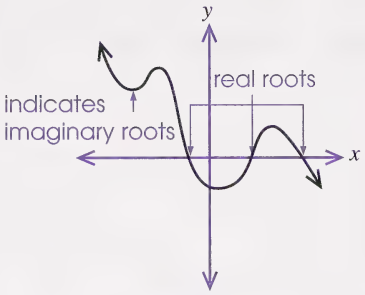
Graph F

- If $n = 3$ (or any odd integer) and a_n is negative, then, by looking at the graph from left to right, the graph begins by falling in the second quadrant and continues moving downward in the fourth quadrant.

In Graph G, the degree of the polynomial is 3 and a_n is negative. In Graph H, the degree of the polynomial is 5, and a_n is negative.



Graph G

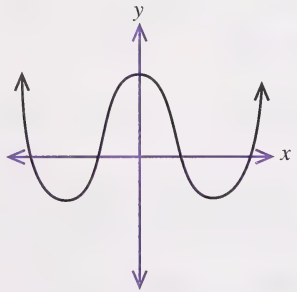


Graph H

The polynomial has three real roots and two imaginary roots.

- If $n = 4$ (or any even integer) and a_n is positive, then, by looking at the graph from left to right, the graph begins by falling to the right in the second quadrant and ends by rising to the right in the first quadrant.

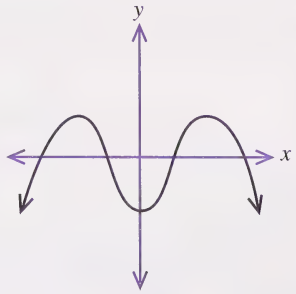
  In Graph I, the degree of the polynomial is 4, and a_n is positive.



Graph I

- If $n = 4$ (or any even integer) and a_n is negative, then, by looking at the graph from left to right, the graph begins by rising in the third quadrant and ends by falling in the fourth quadrant.

  In Graph J, the degree of the polynomial is 4, and a_n is negative.



Graph J

1. Answer the following questions regarding the polynomial $P(x) = (x+3)(x-2)(x+1)$.

a. What is the y-intercept of the graph?

b. What are the zeros of $P(x)$?

c. What is the degree of $P(x)$?

d. Is the initial coefficient a_n positive or negative?

e. Does the graph begin by rising or falling to the right? In which quadrant does it begin?

f. Does the graph end by rising or falling to the right? In which quadrant does it end?

g. Do any of the roots have a multiplicity of more than one?

h. How many real roots and imaginary roots does the polynomial have?

2. Answer the following questions regarding the polynomial $P(x) = -x(x+2)(x-1)(x-3)$.

a. What is the y-intercept of the graph?

b. What are the zeros of $P(x)$?

c. What is the degree of $P(x)$?

d. Is the initial coefficient a_n positive or negative?

e. Does the graph begin by rising or falling to the right? In which quadrant does it begin?

f. Does the graph end by rising or falling to the right? In which quadrant does it end?

g. Do any of the roots have a multiplicity of more than one?

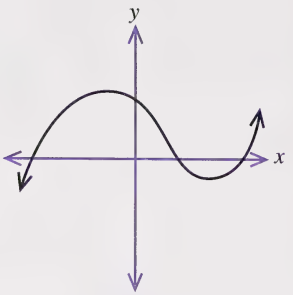
h. How many real roots and imaginary roots does the polynomial have?

Check your answers by turning to the Appendix.

Sketches of Cubic Functions

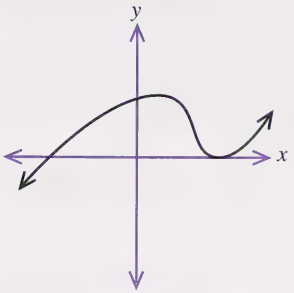
The following four graphs show the possible zero combinations, real and imaginary, for a cubic polynomial when a_n is positive.

In Graph A, the curve crosses the x-axis three times; thus, the polynomial has three real and distinct zeros.



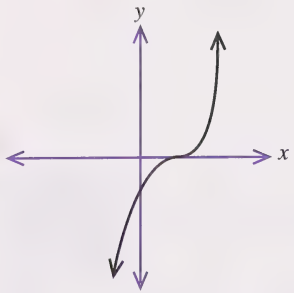
Graph A

In Graph B, the curve crosses the x-axis once and touches the x-axis at another point. The polynomial has three real zeros, two of which are equal.



Graph B

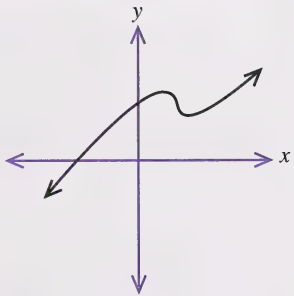
In Graph C, the polynomial has three real and equal zeros. For instance, this graph may represent $P(x)=(x-1)^3$.



Graph C

The multiplicity of the zero 1 is three. The curve will not cross the x-axis as a line, but as a distinct curve.

In Graph D, the curve crosses the x-axis once. The polynomial has one real and two imaginary roots.



Graph D

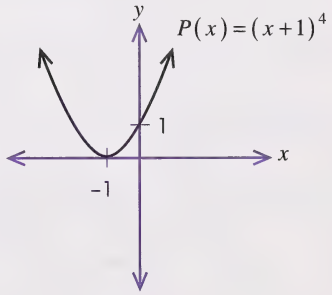
From these four graphs, it is evident that when a_n (the initial coefficient) is **positive**, the graph begins by rising in the third quadrant and finishes by rising in the first quadrant. Thus, the curves must intersect the x-axis at least once. This means that each polynomial must have a minimum of one real root.

If a_n is **negative**, the graph begins by falling in the second quadrant and then moves downward and to the right into the fourth quadrant. Again, each polynomial has at least one real zero since the curve crosses the x-axis at least once.

Sketches of Quartic Functions

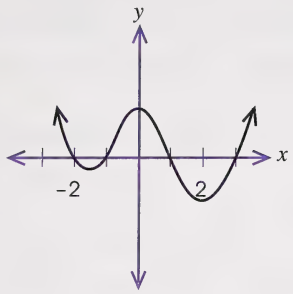
The following sketches show the possible zero combinations (real and imaginary) for a quartic polynomial when a_n is positive.

In Graph A, the curve $P(x)=(x+1)^4$ touches the x-axis at -1. The polynomial has four real equal zeros. Each zero is equal to -1.



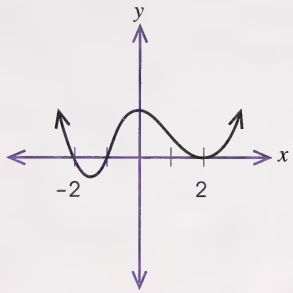
Graph A

In Graph B, the curve intersects the x-axis at four points $(-2, -1, 1, 3)$. Thus, the polynomial has four unequal real zeros.



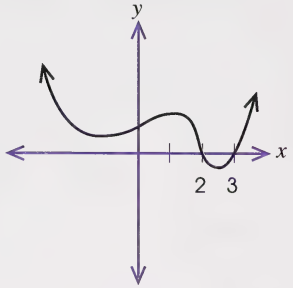
Graph B

In Graph C, the curve crosses the x-axis at -2 and -1. The curve also touches at $x = 2$. The polynomial has four real zeros, two of which are equal.



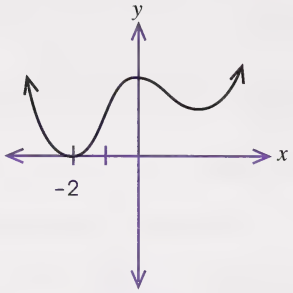
Graph C

In Graph D, the curve crosses the x-axis at 2 and 3. Thus, the polynomial has two unequal real zeros and two imaginary zeros.



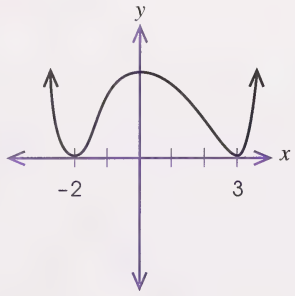
Graph D

In Graph E, the curve touches the x-axis at -2. This polynomial has two equal real zeros and two imaginary zeros.



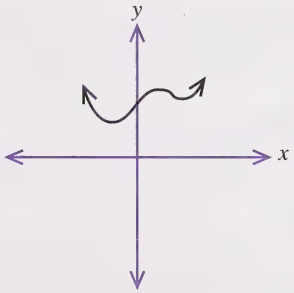
Graph E

In Graph F, the curve touches the x-axis at -2 and 3. This polynomial has two pairs of equal real zeros.

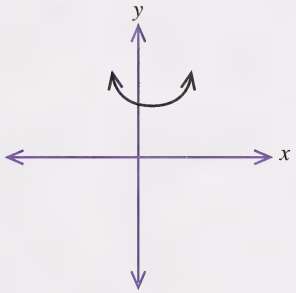


Graph F

In Graphs G and H, the curves do not intersect the x-axis. Thus, the polynomials have four imaginary roots.



Graph G



Graph H

When a_n is **negative**, the graphs of the quartic polynomials will open downwards instead of upwards. The graphs will begin by rising to the right instead of falling to the right.

Look at the summary showing how to sketch graphs of polynomials; then study the examples.

Procedure for Sketching Graphs of Polynomial Functions

So far you have only looked at sketches of polynomial functions. Now you will construct the sketches. Follow these steps:

Step 1: Determine and plot the y-intercept.

Step 2: Determine and plot the zeros of the polynomial.

Step 3: Use the degree and the initial coefficient of the polynomial to obtain the general shape and direction of the curve.

Step 4: Find the multiplicity of the zeros to determine if the curve touches or crosses the x-axis.

Step 5: Find the number of real and nonreal (imaginary) zeros. All real zeros occur when the curve crosses or touches the x-axis. All nonreal zeros occur in pairs.

Example 1

Sketch the graph of $P(x)=(x-2)^3(x+2)^2$.

Solution

Step 1: The y-intercept can be determined by multiplying the constant in each factor.

$$P(x)=(x-2)^3(x+2)^2$$
$$=(x-2)(x-2)(x-2)(x+2)(x+2)$$

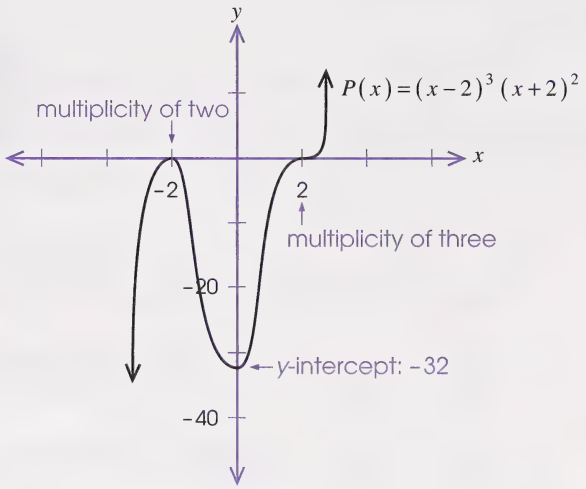
The constant term of this polynomial is $(-2)(-2)(-2)(+2)(+2)=-32$.

Step 2: Since the factors of $P(x)$ are $(x-2)$, $(x-2)$, $(x-2)$, $(x+2)$, and $(x+2)$, the zeros are 2 and -2. Remember that if $x-2$ is a factor, then 2 is a zero since $P(2)=0$.

Step 3: The initial coefficient a_n will be positive since $(x)(x)(x)(x)(x)=+x^5$. This polynomial is of degree 5, and a_n is positive; therefore, the graph begins by rising in the third quadrant and continues moving into the first quadrant.

Step 4: The zero 2 has a multiplicity of three. This multiplicity is an odd number; thus, the curve will pass through $(2,0)$. The zero -2 has a multiplicity of two; thus, the curve will touch the x-axis at $(-2,0)$.

Step 5: There are five real zeros. Where the curve crosses the x-axis at $(2,0)$, there are three real and equal zeros each with a value of 2. Where the curve touches the x-axis at $(-2,0)$, there are two real and equal zeros with a value of -2.



3. Using $P(x)=-(x-2)^2(x-4)$, complete the following questions.

a. What is the y-intercept of the graph?

b. What are the zeros of $P(x)$?

c. What is the multiplicity of each zero?

d. What is the degree of $P(x)$?

e. Is the initial coefficient a_n positive or negative?

f. Does the graph begin by rising or falling to the right? In which quadrant does it begin?

g. Does the graph end by rising or falling to the right? In which quadrant does it end?

h. How many real roots and imaginary roots does the polynomial have?

4. Sketch the graph of $P(x)=(x+3)(x-2)(x+1)$.

5. Using $P(x)=(x+1)^2(x-2)^2$, complete the following questions.

a. What is the y-intercept of the graph?

b. What are the zeros of $P(x)$?

c. What is the multiplicity of each zero?

d. What is the degree of $P(x)$?

e. Is the initial coefficient a_n positive or negative?

f. Does the graph begin by rising or falling to the right? In which quadrant does it begin?

g. Does the graph end by rising or falling to the right? In which quadrant does it end?

h. How many real roots and imaginary roots does the polynomial have?

6. Sketch the graph of $P(x)=-x(x+2)(x-1)(x-3)$.

Check your answers by turning to the Appendix.

You have sketched graphs of polynomial functions in factored form. Now you will sketch graphs of polynomial functions in standard form.

Example 2

Sketch the graph of $P(x)=x^3-7x^2+18x-12$.

Solution

Step 1: To obtain the *y*-intercept, let $x=0$. Thus, the *y*-intercept is -12.

Step 2: The potential rational zeros are ±1, ±2, ±3, ±4, ±6, and ±12.

$$\begin{aligned}P(1)&=(1)^3-7(1)^2+18(1)-12\\&=1-7+18-12\\&=0\end{aligned}$$

Therefore, 1 is a zero and $x-1$ is a factor. Use -1 in the divisor of the synthetic division.

$$\begin{array}{r|rrrr}-1 & 1 & -7 & 18 & -12\\ & & -1 & 6 & -12\\ \hline & 1 & -6 & 12 & \boxed{0}\end{array}$$

Since the remainder is 0, 1 is a zero and $x-1$ is a factor. This checks with the direct substitution.

$$P(x)=(x-1)\left(x^2-6x+12\right)$$

The other two zeros must be obtained from $x^2-6x+12$. Because $x^2-6x+12$ cannot be factored, use the quadratic formula.

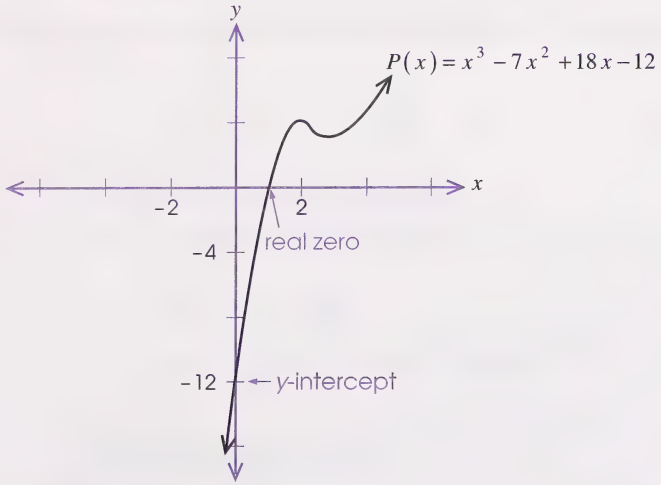
$$\begin{aligned}x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\&=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(12)}}{2(1)}\\&=\frac{6\pm\sqrt{36-48}}{2}\\&=\frac{6\pm\sqrt{-12}}{2}\end{aligned}$$

The zeros are $\frac{6+\sqrt{-12}}{2}$ and $\frac{6-\sqrt{-12}}{2}$. These zeros are nonreal or imaginary values.

Step 3: The polynomial has a degree of 3 and a positive initial coefficient of 1. The graph will start rising in the third quadrant and should continue on into the first quadrant.

Step 4: The real zero, 1, has a multiplicity of one. Thus, the graph crosses the *x*-axis at 1.

Step 5: The polynomial has one real and two nonreal zeros. Since you do not have the skills to determine where the nonreal zeros occur, sketch the general shape and direction of the curve using the *y*-intercept and the real zero.



Example 3

Sketch the graph of the polynomial function
$P(x)=x^5+2x^4-5x^3-18x^2-28x-24$.

Solution

Step 1: To find the y-intercept, let $x=0$. Thus, the y-intercept is -24.

Step 2: The zeros of the polynomial must be determined. The potential rational zeros are ±1, ±2, ±3, ±4, ±6, ±8, ±12, and ±24.

1	1	2	−5	−18	−28	−24
		1	1	−6	−12	−16
	1	1	−6	−12	−16	**−8**

Since the remainder is not 0, then -1 is not a zero.

−1	1	2	−5	−18	−28	−24
		−1	−3	2	20	48
	1	3	−2	−20	−48	**−72**

Since the remainder is not 0, then 1 is not a zero.

2	1	2	−5	−18	−28	−24
		2	0	−10	−16	−24
	1	0	−5	−8	−12	**0**

Since the remainder is 0, then $x+2$ is a factor and -2 is a zero.

$$P(x)=(x+2)\left(x^4-5x^2-8x-12\right)$$

Continue using synthetic division for $x^4-5x^2-8x-12$.

2	1	0	−5	−8	−12
		2	−4	−2	−12
	1	−2	−1	−6	**0**

Since the remainder is 0, then $x+2$ is a factor and -2 is a zero for the second time.

$$P(x)=(x+2)(x+2)\left(x^3-2x^2-x-6\right)$$

Continue using synthetic division for x^3-2x^2-x-6.

$$\begin{array}{r|rrrr} 2 & 1 & -2 & -1 & -6 \\ & & 2 & -8 & 14 \\ \hline & 1 & -4 & 7 & \boxed{-20} \end{array}$$

Since the remainder is not 0, then -2 is not a zero of x^3-2x^2-x-6.

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -1 & -6 \\ & & -2 & 0 & 2 \\ \hline & 1 & 0 & -1 & \boxed{-8} \end{array}$$

Since the remainder is not 0, then 2 is not a zero.

$$\begin{array}{r|rrrr} 3 & 1 & -2 & -1 & -6 \\ & & 3 & -15 & 42 \\ \hline & 1 & -5 & 14 & \boxed{-48} \end{array}$$

Since the remainder is not 0, then -3 is not a zero.

$$\begin{array}{r|rrrr} -3 & 1 & -2 & -1 & -6 \\ & & -3 & -3 & -6 \\ \hline & 1 & 1 & 2 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x-3$ is a factor and 3 is a zero.

$$P(x)=(x+2)(x+2)(x-3)\left(x^2+x+2\right)$$

The remaining two zeros must be obtained from x^2+x+2. Since x^2+x+2 is not factorable over the integers, the quadratic formula must be used.

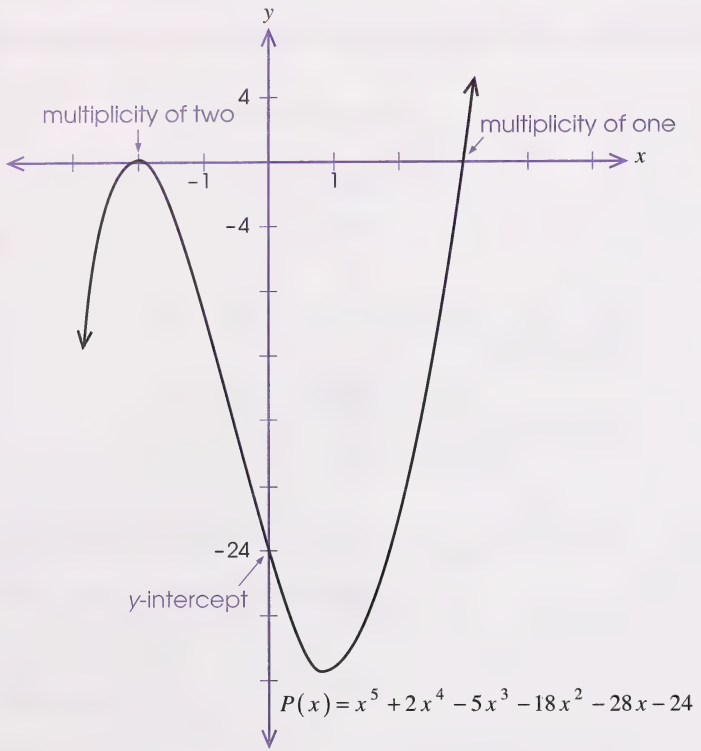
$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-1\pm\sqrt{1^2-4(1)(2)}}{2(1)} \\ &= \frac{-1\pm\sqrt{-7}}{2} \end{aligned}$$

The remaining two zeros are nonreal (imaginary).

Step 3: Since the degree of the polynomial is 5 and the initial coefficient is positive, the graph begins by rising in the third quadrant and ends by rising in the first quadrant.

Step 4: The zero 3 has a multiplicity of one; and the zero -2 has a multiplicity of two. The curve crosses the x-axis at $(3,0)$ and touches the x-axis at $(-2,0)$.

Step 5: The polynomial has three real and two nonreal roots. The three real roots are -2, -2, and 3. With this information, a sketch of the polynomial can now be drawn.



Example 4

Sketch the graph of the polynomial function $P(x)=x^4+3x^3-3x^2-11x-6$.

Solution

Step 1: To determine the y-intercept, let $x=0$. Thus, the y-intercept is -6.

Step 2: The potential rational zeros are ±1, ±2, ±3, and ±6. Now use synthetic division to determine the factors and the zeros.

1	1	3	−3	−11	−6
		1	2	−5	−6
	1	2	−5	−6	0

Since the remainder is 0, then $x+1$ is a factor and -1 is a zero.

$$P(x)=(x+1)\left(x^3+2x^2-5x-6\right)$$

Continue using synthetic division for the quotient polynomial x^3+2x^2-5x-6.

1	1	2	−5	−6
		1	1	−6
	1	1	−6	0

Since the remainder is 0, then $x+1$ is a factor and -1 is a zero for a second time.

$$P(x)=(x+1)(x+1)\left(x^2+x-6\right)$$

To determine the remaining two zeros, factor x^2+x-6.

$$x^2+x-6=(x-2)(x+3)$$

Let $(x-2)(x+3)=0$.

$x-2=0$ or $x+3=0$

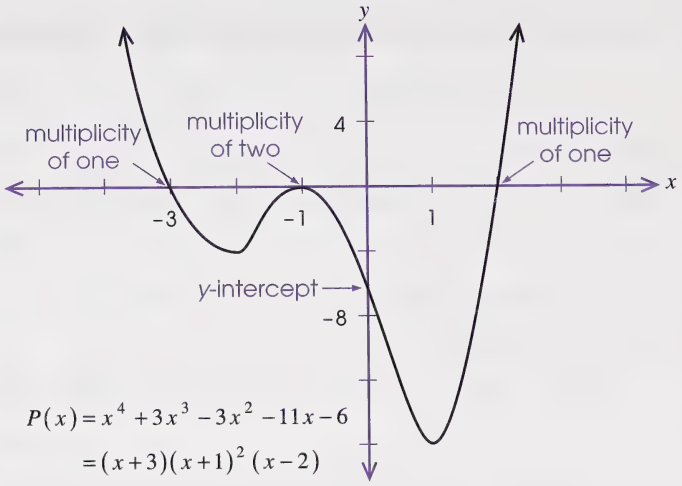
$x=2$ $\qquad$ $x=-3$

The zeros are 2, -3, and -1 (multiplicity of two).

Step 3: Since the degree of the polynomial is 4 and the initial coefficient is positive, the graph begins by falling to the right in the second quadrant and ends by rising to the right in the first quadrant.

Step 4: The zeros 2 and -3 both have a multiplicity of one; thus, the curve crosses the x-axis at $(2,0)$ and $(-3,0)$. The zero -1 has a multiplicity of two; thus, the graph touches at $(-1,0)$.

Step 5: The polynomial has four real roots. Two of the roots are equal. Now you can sketch the polynomial.



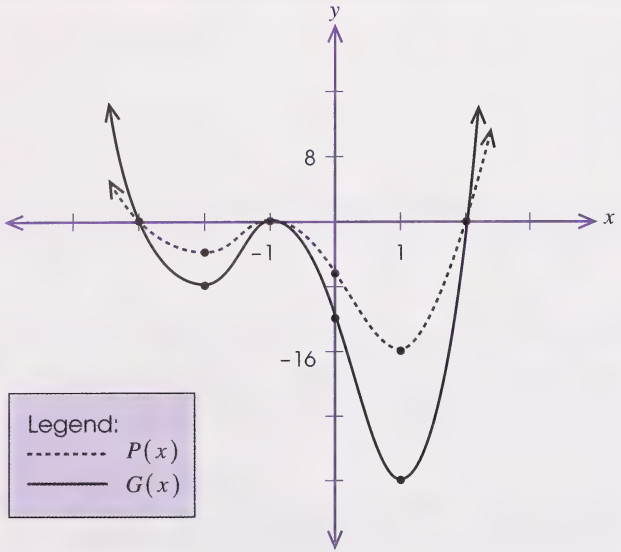
Example 5

Sketch the graph $G(x)=2x^4+6x^3-6x^2-22x-12$.

Solution

In Example 4, the polynomial was $P(x)=x^4+3x^3-3x^2-11x-6$. If you compare $G(x)=2x^4+6x^3-6x^2-22x-12$ to $P(x)$, you will see that $G(x)=2P(x)$. When the graphs of these polynomials are drawn on the same axes, the zeros for both polynomials are the same. The y-intercept and other y-values of $G(x)$ are twice the values of $P(x)$.

The sketch of both graphs is shown.



From the theory and the sketch, it is evident that whenever a polynomial has a constant factor, this factor changes the shape of the plotted graph by increasing the y-values by a multiple of the constant factor.

Using Zap-a-Graph™

If you have access to a Macintosh computer, you may use the graphing application Zap-a-Graph™ to generate graphs of some polynomial functions.

The following information will help you graph polynomial functions, like $G(x)=2x^4+6x^3-6x^2-22x-12$, using Zap-a-Graph™.

After opening the program, follow these steps:

Step 1: Select "Quartic" from the Define menu. This will open a window that will permit you to enter the coefficients of the function.

Step 2: Enter the coefficients in each of the boxes for "a" to "e". You can move between the boxes by either clicking on the box or by pressing the tab key.

Step 3: Click on Plot, or press the Return key.

At this point the graph will be displayed on the screen.

Yes, you have the graph; but the critical points of the graph are not all visible. You can get a better display of the graph by following these steps:

Step 1: Pull down the Grid menu and select "Scale". A window will open that will allow you to set the scale on both axes. The larger the values entered here, the smaller the graph will appear on the screen. The original values for the scale will be 5 for both scales.

Step 2: Set the value of the x-tics to 1 and the value of the y-tics to 10.

Step 3: Click on OK, or press the Return key.

The graph will now fill most of the screen and you can see all of the critical points.

You can now use some of the features of the program to find some vital information about the graph. For example, do the following:

- Select "Analyze" from the Options menu. On the screen you see the message "y-intercept: -12." Click the screen and the message will disappear.

- Select "Table of Values" from the Option menu. A table of values will appear on the screen giving just some of the ordered pairs from the function. With the options available, you can select different ordered pairs. Close the window by clicking the little box in the top left corner.

- Select "Root Search" from the Options menu. Here, a window will appear. Select two values of x that are on the opposite sides of the graph, such as -5 and 5. Press the Return key. The program will tell you that there is a root at $x=-3$. Change the first x to -2 and the second x to 0. The program will now tell you that a root is at $x=-1$. Change the first x value to 0 and the second x to 5. The program will tell you there is a root at $x=2$. Click cancel and the window will disappear.

Try some of the other options and see what happens. For instance, to erase the graph, select "Everything" from the Erase menu. The screen will be cleared. You are now ready to graph your next function.

Here are steps for graphing a function.

Step 1: Select the appropriate type of graph from the Define menu.

Step 2: Enter the appropriate coefficients. Use the tab key to move between coefficients or click on the desired box.

Step 3: Click on Plot, or press the Return key to graph the function.

For practice, do Examples 2 and 3 in this activity using Zap-a-Graph™, and compare your solutions.

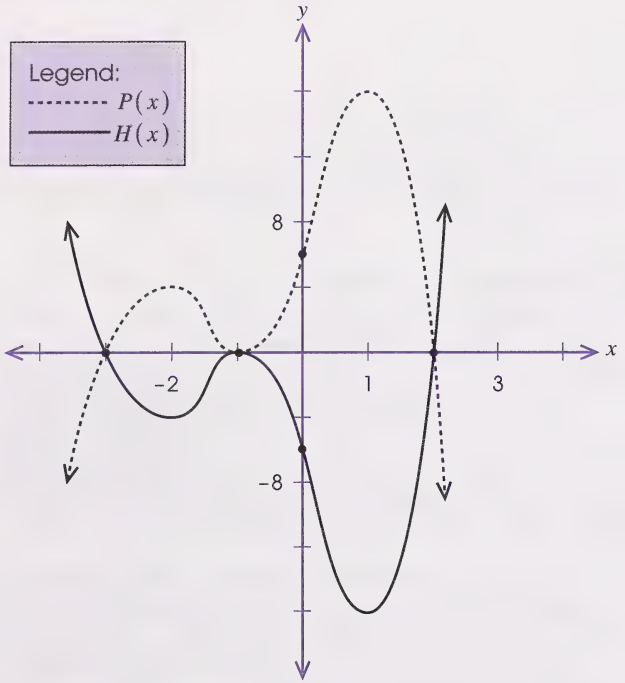
Example

Sketch the graph of the polynomial
$H(x)=-x^4-3x^3+3x^2+11x+6.$

Solution

In Example 4, the polynomial was
$P(x)=x^4+3x^3-3x^2-11x-6$. If you compare
$H(x)=-x^4-3x^3+3x^2+11x+6$ to $P(x)$, you will see that
$H(x)=-P(x)$. When the graphs of these polynomials are drawn on the same axes, the zeros for both polynomials are the same.

Compare the sketch of both graphs.



If two polynomials have opposite signs for each term, the flow of direction is reversed. The graphs are symmetrical about the x-axis.

7. Using $P(x)=x^2-x-6$, complete the following questions.

a. What is the y-intercept of the graph?

b. What are the factors of x^2-x-6?

c. What are the zeros of $P(x)$?

d. What is the degree of $P(x)$?

e. Is the initial coefficient a_n positive or negative?

f. Does the graph begin by rising or falling to the right? In which quadrant does it begin?

g. Does the graph end by rising or falling to the right? In which quadrant does it end?

h. How many real roots does the polynomial have?

i. Sketch the graph of the polynomial.

8. Sketch the graphs of the following polynomials.

a. $P(x)=-3x^3+8x^2+5x-6$

b. $P(x)=4x^3-3x^2-16x+12$

c. $P(x)=2x^4+x^3-17x^2-16x+12$

d. $P(x)=-2x^4-x^3+15x^2+8x-20$

e. $P(x)=4x^4-4x^3-11x^2+6x+9$

9. Generate the graphs in question 8 using a graphing calculator or Zap-a-Graph™. Compare each graph with the preceding sketch.

Check your answers by turning to the Appendix.

To draw a graph, it is helpful to find the zeros of the polynomial. The degree and coefficient of the term with the highest degree affect graphs of polynomial functions.

Activity 3: Deriving Integral Polynomial Functions

So far in this module you have been given the equation of a polynomial function and you have found its zeros. You have also found ordered pairs that satisfy the equation.

In this activity you will derive the equation of an integral polynomial function given its zeros, or given its zeros and an ordered pair that satisfies it. You will also derive the equation of a polynomial function given its zeros and any other information that will uniquely define it.

Example 1

Determine the integral polynomial of degree 3 that has the zeros 2, -1, and $\frac{1}{4}$.

Solution

If 2 is a zero, then $x-2$ is a factor. If -1 is a zero, then $x+1$ is a factor. Since $\frac{1}{4}$ is a zero, then $x-\frac{1}{4}$ is a factor; but $x-\frac{1}{4}$ is a rational polynomial. If $x-\frac{1}{4}$ is multiplied by four, then it becomes $4x-1$ (an integral polynomial with a zero of $\frac{1}{4}$). These three integral factors can be multiplied to form the required polynomial.

$$\therefore P(x) = (x-2)(x+1)(4x-1)$$
$$= 4x^3 - 5x^2 - 7x + 2$$

There are many integral polynomials that can have the same zeros as Example 1. For instance, another polynomial having the same zeros could be the following:

$$P(x) = -1(x-2)(x+1)(4x-1)$$
$$= -4x^3 + 5x^2 + 7x - 2$$

$P(x) = 4x^3 - 5x^2 - 7x + 2$ could be multiplied by any real number to produce polynomials which would have the same zeros.

Example 2

Determine the integral polynomial of degree 3 that has the zeros 2, -1, and $\frac{1}{4}$ and a remainder of -18 when divided by $x-1$.

Solution

Method 1

From Example 1, $P(x)=4x^3-5x^2-7x+2$ is one polynomial that has the given zeros. Let $g(x)$ be the required polynomial that will have a remainder of -18 when divided by $x-1$. Using the Remainder Theorem, $g(1)=-18$. Since $P(1)=-6$ and $g(1)=-18$, then $P(x)$ is not equal to $g(x)$.

Thus, $P(x)$ must be changed to obtain $g(x)$ without changing the zeros. The zeros are dependent on the factors; thus, the factors cannot be changed.

If $P(1)=-6$ and $g(1)=-18$, then it seems reasonable that three times $P(x)$ gives $g(x)$ since $3(-6)=-18$.

Thus, $g(x)=3P(x)=12x^3-15x^2-21x+6$.

Check

$$\begin{aligned} g(1) &= 12(1)^3-15(1)^2-21(1)+6 \\ &= 12-15-21+6 \\ &= -18 \end{aligned}$$

Therefore, $g(x)=12x^3-15x^2-21x+6$ is the required integral polynomial of degree 3 that has the zeros 2, -1, $\frac{1}{4}$ and a remainder of -18 when divided by $x-1$.

Method 2

A polynomial with zeros of 2, -1, and $\frac{1}{4}$ is given by $g(x)=k(x-2)(x+1)(4x-1)$.

A remainder of -18 is produced when the divisor is $x-1$; thus, $g(1)=-18$.

$$\begin{aligned} -18 &= k(1-2)(1+1)[4(1)-1] \\ -18 &= k(-1)(2)(3) \\ -18 &= -6k \\ k &= 3 \end{aligned}$$

$$\begin{aligned} \therefore\ g(x) &= 3(x-2)(x+1)(4x-1) \\ &= 12x^3-15x^2-21x+6 \end{aligned}$$

If a polynomial is multiplied or divided by a constant, then the zeros will not change.

Multiplying or dividing by a constant will change the graph of the polynomial, but not the zeros.

In order to uniquely define an integral polynomial, you must be given all the zeros as well as some other information. This other information could be the following:

- a remainder when the polynomial is divided by a divisor

  For example, a remainder of -18 when the divisor is $x-1$ (see Example 2).

- a y-intercept

  For example, a y-intercept of -4. This means that $P(0)=-4$. Determine the polynomial from the product of the integral factors; then substitute 0 for the variables in the polynomial and see what remainder is obtained. For instance, if the remainder is 2, then the polynomial must be multiplied by -2 in order to produce the unique polynomial.

- an initial coefficient

  For instance, an initial coefficient of -10. If the product of the integral factors produces a polynomial which has an initial coefficient of -2, then the unique polynomial is obtained when the first polynomial is multiplied by 5.

- P at a certain value which produces a value for the polynomial

  For example, $P(1)=-36$. Substitute 1 into the polynomial produced by the product of the integral factors. If the value of this polynomial is 9 when 1 is substituted, then this polynomial must be multiplied by -4 in order to obtain the unique polynomial.

1. a. Determine the integral polynomial of degree 3 where the zeros are 1, $\frac{1}{3}$, and $-\frac{1}{4}$ and $P(-1)=48$.

b. Determine the integral polynomial of degree 3 where the zeros are -2, $\frac{1}{2}$, and $-\frac{3}{2}$ and $P(1)=30$.

2. a. Determine the integral polynomial of degree 4 where the zeros are -1, 2, $-\frac{1}{2}$, and $\frac{2}{3}$ and $P(1)=6$.

b. Determine the fourth-degree integral polynomial which has zeros -2, and $-\frac{1}{3}$, each of multiplicity two, where $P(1)=288$.

3. a. Determine the third-degree integral polynomial with zeros $\frac{1}{3}$, $\frac{3}{2}$, and -2 that has a y-intercept of -18.

b. Determine the third-degree integral polynomial with zeros $\frac{1}{2}$ and -1 (multiplicity of two). This polynomial has a y-intercept of 5.

4. a. Determine the fourth-degree integral polynomial with zeros -1, 2, $\frac{1}{3}$, and $-\frac{1}{2}$ such that the initial coefficient is -18.

b. Determine the fourth-degree integral polynomial with zeros 1 (multiplicity of 2), $\frac{2}{3}$, and $-\frac{1}{4}$ such that the initial coefficient is 24.

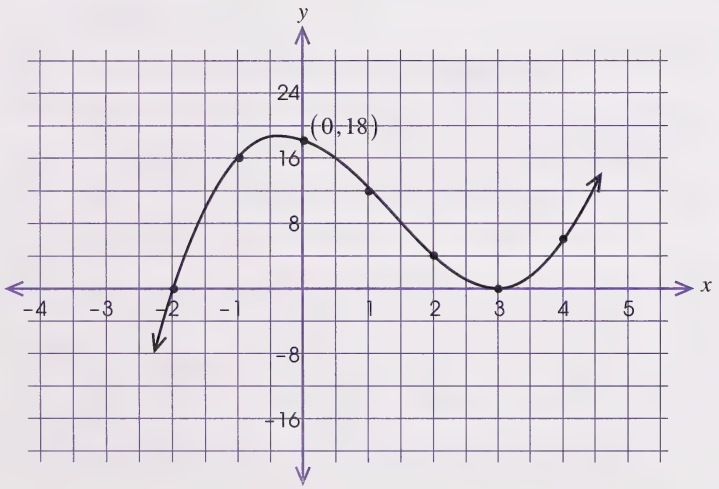
Check your answers by turning to the Appendix.

Activity 4: Determining the Polynomial Function of a Given Graph

A polynomial function can be determined if the graph of the polynomial function and the degree of the polynomial are given.

Example 1

The following is a graph of a polynomial function. Determine the polynomial function of degree 3 which represents the graph. For more theory on interpreting the degree of a graph, go to the Enrichment at the end of this section.



Solution

The graph starts rising to the right in the third quadrant. It changes direction twice and continues rising to the right in the first quadrant. This graph is of degree 3; thus, the initial coefficient is positive.

The real zeros are -2 and 3. Since the graph touches the x-axis at 3, then 3 has a multiplicity of 2.

If the zeros are known, then the factors can be obtained. Since -2 is a zero, then $x+2=0$ and the factor is $x+2$. If 3 is a zero, then $x-3=0$ and the factor is $x-3$. Since 3 has a multiplicity of two, then there will be two factors of $x-3$.

You must tentatively assume that $P(x)=a(x+2)(x-3)(x-3)$ is the form of the function. Choose a point on the curve not on the x-axis. Since the y-intercept is 18, $P(0)=18$.

$$18=a(0+2)(0-3)(0-3)$$
$$18=18a$$
$$a=1$$

To obtain the polynomial, multiply the factors by themselves.

$$\begin{aligned} P(x) &= 1(x+2)(x-3)(x-3) \\ &= \left(x^2-x-6\right)(x-3) \\ &= x^3-4x^2-3x+18 \end{aligned}$$

You should also be able to determine a polynomial function if you are given the zeros, another ordered pair, and the degree of the function.

Example 2

Determine the third-degree polynomial function which touches the x-axis at $(-3,0)$ and intersects the x-axis at $(2,0)$. A point on the curve is $(-2,8)$.

Solution

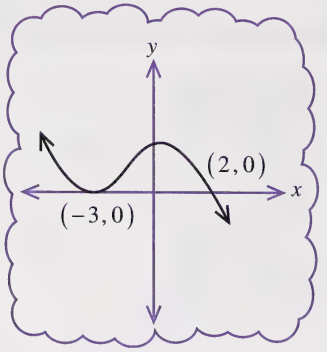
Since the graph touches the x-axis at $(-3,0)$, then the zero -3 has a multiplicity of two. Since -3 is a zero, then $x+3=0$ and the factor is $x+3$. There will be two factors of $x+3$ since the multiplicity is two. The curve intersects the x-axis at $(2,0)$. Since 2 is a zero, then $x-2=0$ and the factor is $x-2$.

> Sometimes the phrase **is tangent to the x-axis** is used instead of **touches the x-axis**.

Assume that $P(x)=a(x-2)(x+3)(x+3)$. Since $P(-2)=8$, solve for a.

$$8=a(-2-2)(-2+3)(-2+3)$$
$$8=a(-4)(1)(1)$$
$$a=-2$$

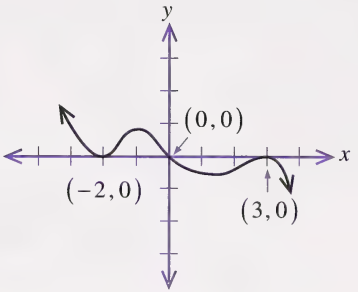
$$\therefore P(x)=-2(x-2)(x+3)(x+3)$$
$$=-2\left(x^2+x-6\right)(x+3)$$
$$=-2\left(x^3+4x^2-3x-18\right)$$
$$=-2x^3-8x^2+6x+36$$



If you are given a graph of a polynomial function showing all the real zeros, you should be able to determine the polynomial function. Look at a condition where a point on the curve is not known.

Example 3

From the information on the graph, determine the polynomial function.



Solution

The graph touches the x-axis at $(-2,0)$; thus, the zero -2 has a multiplicity of two. The two factors will be $(x+2)$ and $(x+2)$.

The graph crosses the x-axis at $(0,0)$; thus, the zero 0 has a multiplicity of one. The one factor is x.

The graph touches the x-axis at $(3,0)$; thus, the zero 3 has a multiplicity of two. The two factors will be $(x-3)$ and $(x-3)$.

The degree of this polynomial must be equal to the sum of the multiplicities. So, the degree is $(2+1+2)=5$.

Since the degree is an odd integer and the graph begins by falling to the right in the second quadrant and continues into the fourth quadrant, the initial coefficient is negative.

Assume that $P(x)=ax(x+2)(x+2)(x-3)(x-3)$, and multiply the factors.

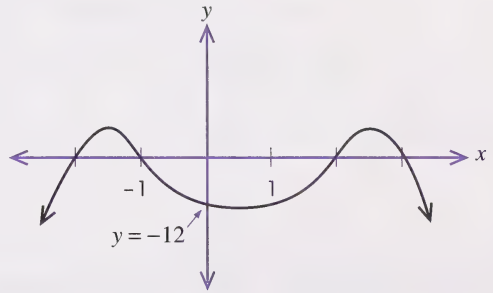
$$\begin{aligned} P(x) &= a(x)(x+2)(x+2)(x-3)(x-3) \\ &= a(x)\left(x^2+4x+4\right)\left(x^2-6x+9\right) \\ &= a\left(x^5-2x^4-11x^3+12x^2+36x\right) \end{aligned}$$

Since no point on the curve is given, you cannot find a definite value for a. Therefore, a is any value less than zero.

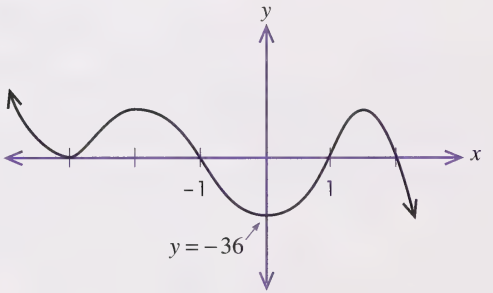
1. The polynomials for each graph have real zeros. For each sketch, state the following:

- the zeros and their multiplicities
- the degree of the polynomial
- the polynomial function

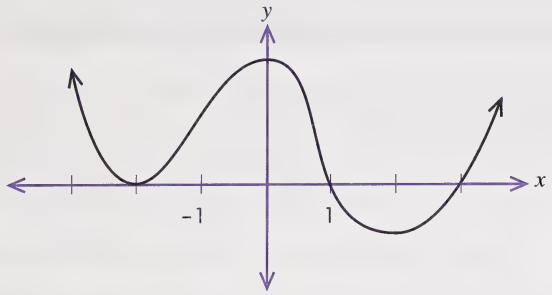
a.



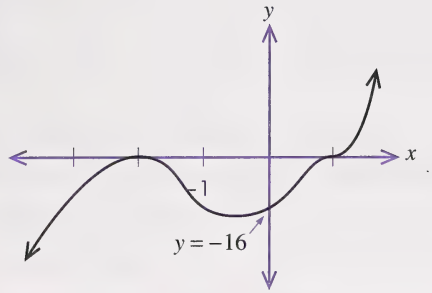
b.



c.



d.



2. Determine the third-degree polynomial function which crosses the x-axis at $(-2, 0)$ and touches the x-axis at $(1, 0)$. The y-intercept is 2.

3. Determine the fourth-degree polynomial function which touches the x-axis at $(-1, 0)$ and crosses the x-axis at $(2, 0)$ and $(3, 0)$. The polynomial also passes through the point $(1, -8)$.

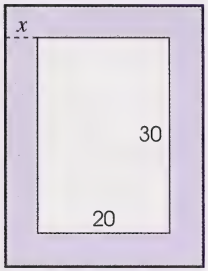
Check your answers by turning to the Appendix.

Look at the next example of a real-life situation.

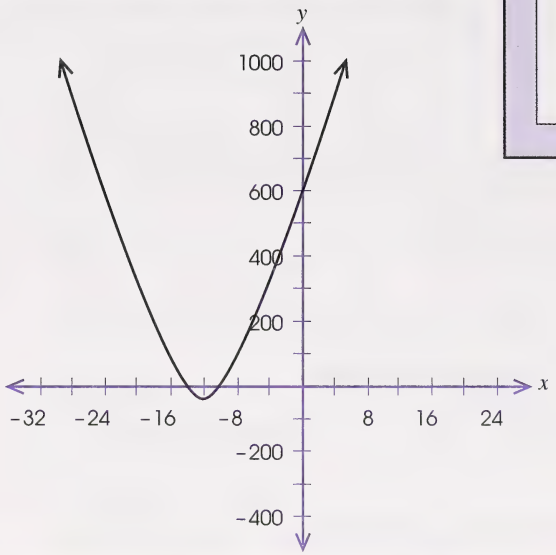
Example 4

A picture measuring 30 cm by 20 cm is surrounded by a frame of uniform width (x cm). What function describes the area of the framed picture? Graph the function and determine when the framed picture will be double the original area.

Solution



$$y = (20 + 2x)(30 + 2x)$$
$$= 4x^2 + 100x + 600$$



Original Area $= 30\ \text{cm} \times 20\ \text{cm}$

$= 600\ \text{cm}^2$

$4x^2 + 100x + 600 = 2(600)$

$4x^2 + 100x + 600 = 1200$

$4x^2 + 100x - 600 = 0$

$x^2 + 25x - 150 = 0$

$(x - 5)(x + 30) = 0$

$x - 5 = 0$ or $x + 30 = 0$

$x = 5$ $\qquad$ $x = -30$ (impossible)

When $x = 5$ cm, the framed picture will be double the original area.

4. A rectangular mirror is surrounded by a uniform frame measuring 12 cm by 8 cm. Let x represent the width of the frame.

a. Write a function which represents the area of the mirror.

b. Graph the function.

c. Why is it impossible to have a frame 4 cm wide?

d. What will be the width of the frame that will enclose a mirror with an area of $32\ \text{cm}^2$?

e. What is the domain of this function?

5. A sheet of metal 20 cm wide is bent to construct an eavestrough.

a. Write the cross-sectional area as a function of x, the height of the eavestrough.

b. Graph this function.

c. Is the domain of this function restricted in any way?

d. What is happening to the cross-sectional area?

e. What do the x-intercepts of the graph indicate in real terms?

6. $d = t^3 + t^2 + 8$ represents the position of an object travelling in a straight line at time t (in seconds). Distance d (in metres) is measured from a fixed point (or origin).

a. Graph this function.

b. What does the t-intercept represent?

c. Where is the object at $t = 2$?

d. When is the object 158 m from the origin?

Check your answers by turning to the Appendix.

You have developed an understanding that graphs describe relations among variables in many real-world situations.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

You may find it helpful to view the video titled *Graphing Polynomial Functions*. This is Program 5 of the *Graphing Mathematical Concepts* series, ACCESS Network. The program will help to reinforce the concepts of degree and number of roots, and of degree and the shape of the graph. In addition, multiplying a polynomial by -1 and its effect on the graph is discussed. This video is available from Learning Resources Distributing Centre.

Do the following questions.

For questions 1 to 13, fill in the blanks.

1. Ordered pairs of real numbers can be plotted on a rectangular coordinate system, which is also called a _____ coordinate system.

2. To draw a graph of a polynomial, use a _____; then plot the points represented by each _____.

3. The x-intercepts are the _____ of the polynomial.

4. To determine the y-intercept, let _____.

5. The degree of a polynomial is at least _____ than the number of changes of direction in the graph.

6. The zeros of a polynomial may be real or _____.

7. Real zeros occur at the points where the polynomial _____ or _____ the x-axis.

8. The _____ of a zero is the number of times that the same zero occurs for a polynomial.

9. If the _____ of the zero is even, then the graph _____ the x-axis. If the _____ is odd, then the graph _____ the x-axis.

10. $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$.
If $n = 3$ (or any odd integer) and a_n is positive, then the graph begins by _____ in the _____ quadrant and ends by _____ in the _____ quadrant.

11. If $n = 3$ (or any odd integer) and a_n is negative, then the graph begins by _____ to the right in the _____ quadrant and ends by _____ to the right in the _____ quadrant.

12. If $n = 4$ (or any even integer) and a_n is positive, then the graph begins by _____ to the right in the _____ quadrant and ends by _____ to the right in the _____ quadrant.

13. If $n = 4$ (or any even integer) and a_n is negative, then the graph begins by _____ in the _____ quadrant and ends by _____ to the right in the _____ quadrant.

14. Complete the following questions concerning the polynomial $P(x) = -(x+3)(x+2)(x-1)^2$.

a. What is the y-intercept of the graph?

b. What are the zeros of $P(x)$?

c. What is the multiplicity of each zero?

d. What is the degree of $P(x)$?

e. Is the initial coefficient a_n positive or negative?

f. Does the graph begin by rising or falling to the right? In which quadrant does it begin?

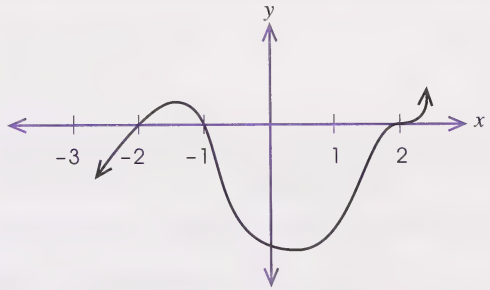
g. Does the graph end by rising or falling to the right? In which quadrant does it end?

h. How many real roots and imaginary roots does the polynomial have?

i. Sketch the graph of the polynomial.

15. Sketch the graph of $P(x) = -x^3 - x^2 + 8x + 12$.

16. In the following sketch, the polynomial has only real zeros.



a. State the zeros and their multiplicities.

b. State the degree of the polynomial.

c. Determine the polynomial function in factored form if the leading coefficient is 1.

d. Determine the integral polynomial if the initial coefficient is 2.

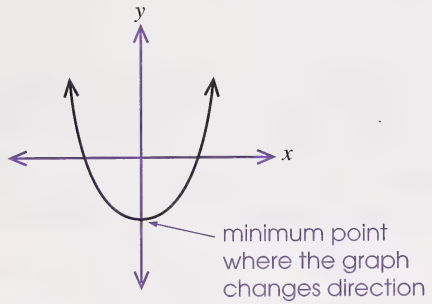
17. Determine the third-degree polynomial which crosses the x-axis at $(-3, 0)$ and is tangent to the x-axis at $(2, 0)$. The curve also passes through the point $(1, -4)$.

When the curve is tangent to the x-axis, it means the curve touches the x-axis.

Check your answers by turning to the Appendix.

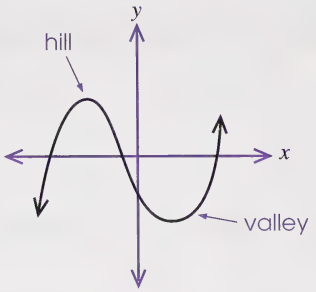
Enrichment

In this section, under the heading **Factors Affecting Graphs of Polynomial Functions**, it is stated that the degree of a polynomial is at least one more than the number of changes in direction in the graph. For example, consider the quadratic polynomial in Graph A. This graph has one change in direction. From left to right, the graph is moving downwards to a minimum; then the graph increases. Since the graph changes direction once, the degree of the polynomial must be 2 (one more than the number of changes). The point where the change occurs can be referred to as a valley.



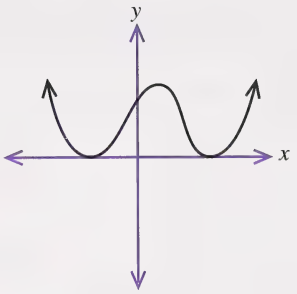
Graph A

In Graph B, the polynomial shown is of degree 3. Note that there are two changes in direction. The first change can be referred to as a hill, and the second change can be referred to as a valley. One more than the number of changes gives three, which is the degree of the polynomial.



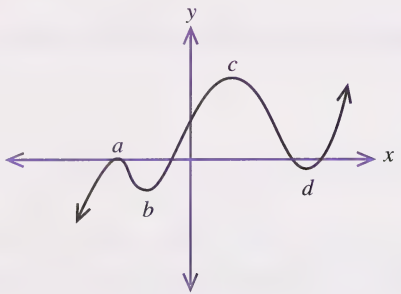
Graph B

In Graph C, the polynomial shown is of degree 4. Note that there are three changes in direction, which are represented by two valleys and one hill. One more than the number of changes gives four, which is the degree of the polynomial.



Graph C

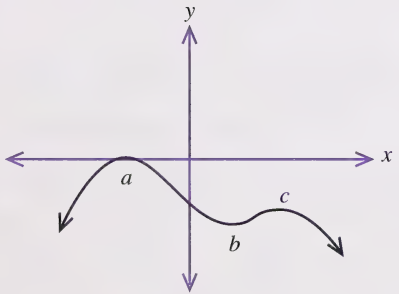
In Graph D, the polynomial shown is of degree 5. There are four changes in direction. Two changes in direction are represented by hills a and c. The other changes in direction are represented by valleys b and d. One more than the number of changes gives five, which is the degree of the polynomial.



Graph D

In the preceding graphs, the zeros are all real since the curve either crosses or touches the x-axis. A polynomial may have some nonreal zeros, or all the zeros may be nonreal.

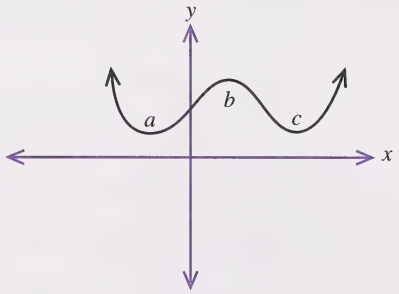
In Graph E, the polynomial shown is of degree 4. There are three changes in direction. Two changes in direction are represented by hills a and c. Another change in direction is represented by valley b. One more than the number of changes gives four, which is the degree of the polynomial. This polynomial has two real equal roots and two nonreal roots.



Graph E

Remember that nonreal roots occur in pairs.

In Graph F, the polynomial shown is of degree 4. There are three changes in direction. Two changes in direction are represented by valleys a and c. One change in direction is represented by hill b. One more than the number of changes gives four, which is the degree of the polynomial. This polynomial has four nonreal roots.

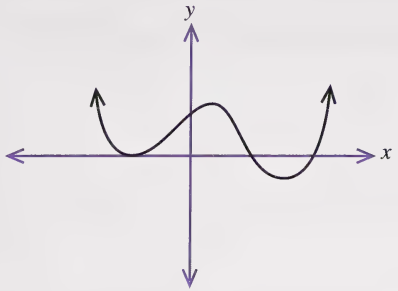


Graph F

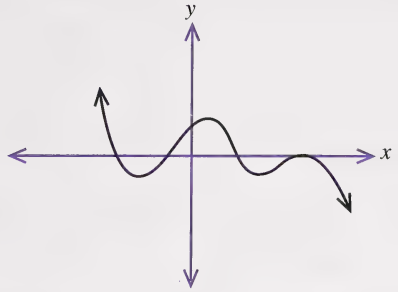
For questions 1 to 6, the sketch of a polynomial is given. For each sketch, do the following:

- Determine the number of valleys and number of hills.
- Determine the degree of the polynomial.
- Determine the number of real zeros and the number of nonreal zeros.
- Determine the sign of the initial coefficient of the polynomial.

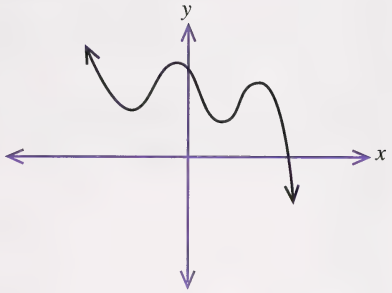
1.



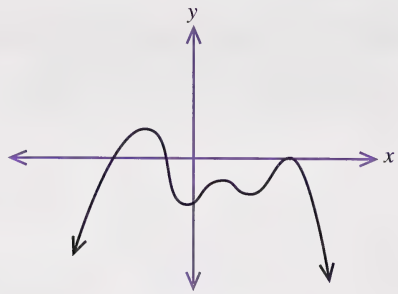
2.



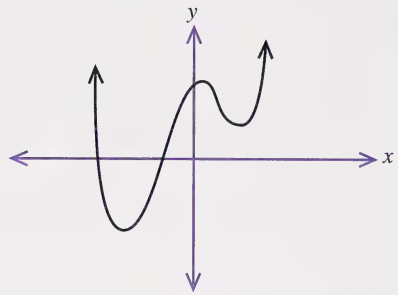
3.



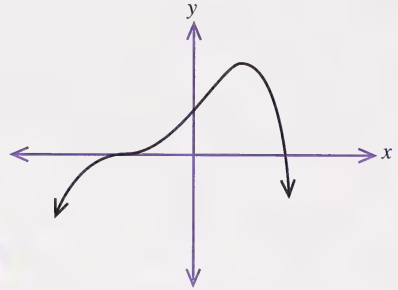
4.



5.



6.



Check your answers by turning to the Appendix.

Previously, you were shown a method for determining approximations for irrational zeros. The following example will illustrate how to obtain the polynomial function when given the zeros (rational and irrational), another ordered pair, and the degree of the function.

Example

Determine the third degree polynomial which has zeros 2, $1+\sqrt{3}$, and $1-\sqrt{3}$. The curve passes through the point $(1,-6)$.

Solution

If 2 is a zero, then $x-2$ is a factor.

If $1+\sqrt{3}$ is a zero, then $x-\left(1+\sqrt{3}\right)$ is a factor.

If $1-\sqrt{3}$ is a zero, then $x-\left(1-\sqrt{3}\right)$ is a factor.

$$P(x)=(x-2)\left[x-\left(1+\sqrt{3}\right)\right]\left[x-\left(1-\sqrt{3}\right)\right]$$

The following multiplication shows how $\left[x-\left(1+\sqrt{3}\right)\right]$ is multiplied by $\left[x-\left(1-\sqrt{3}\right)\right]$.

$$\begin{array}{r} x-\left(1+\sqrt{3}\right) \\ x-\left(1-\sqrt{3}\right) \\ \hline -\left(1-\sqrt{3}\right)x-2 \\ x^2-\qquad \left(1+\sqrt{3}\right)x \\ \hline x^2-\left(1-\sqrt{3}\right)x-\left(1+\sqrt{3}\right)x-2 \end{array}$$

$$\begin{aligned} &= x^2-x+\sqrt{3}x-x-\sqrt{3}x-2 \\ &= x^2-2x-2 \end{aligned}$$

Thus, $P(x)=(x-2)\left(x^2-2x-2\right)$

$$= x^3-4x^2+2x+4$$

Check if this polynomial satisfies the ordered pair $(1,-6)$.

$$\begin{aligned} P(x) &= (1)^3-4(1)^2+2(1)-4 \\ &= 1-4+2+4 \\ &= 3 \end{aligned}$$

The polynomial has a value of 3 when x is 1, but the required value is -6. Since $-2\times 3=-6$, multiply the polynomial by -2.

$$\begin{aligned} P(x) &= -2\left(x^3-4x^2+2x+4\right) \\ &= -2x^3+8x^2-4x-8 \end{aligned}$$

7. Determine the third degree polynomial which has zeros -3, $1+\sqrt{5}$, and $1-\sqrt{5}$. The curve passes through the point $(1,-10)$.

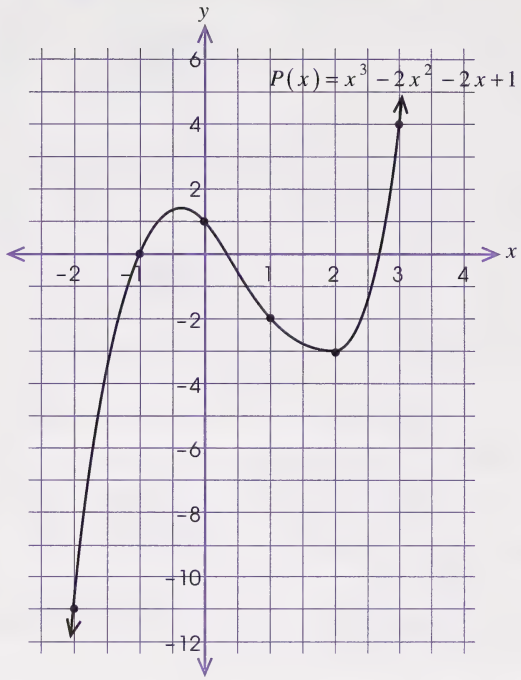
Check your answers by turning to the Appendix.

Identifying All Zeros of a Polynomial Function

Irrational zeros of a polynomial function of degree greater than two cannot be found exactly, but approximations for the zeros can be determined. Approximations of the irrational zeros can be obtained from a graph of the polynomial. The following is a graph of the polynomial $P(x)=x^3-2x^2-2x+1$. In order to draw the graph, a table of values must be used.

An irrational number is a nonrational number. It is a real number which cannot be expressed as the quotient of integers. An irrational number has infinite decimals that do not repeat.

x	-2	-1	0	1	2	3
$P(x)$	-11	0	1	-2	-3	4



Since the graph of the polynomial intersects the x-axis at three different points, the polynomial has three real zeros. One zero, -1, is a rational number. The other two zeros must be irrational since the potential rational zeros are ± 1. From the graph, one irrational zero is between 0 and 0.5 and the other is between 2.5 and 3.

Note: Always determine any rational zeros before attempting to find the approximate irrational zeros.

If $P(x_1)$ is greater than zero and $P(x_2)$ is less than zero, then there is one x between x_1 and x_2 such that $P(x)=0$.

Examine the table of values. Note that when $x=0$, $P(x)$ is 1; and when $x=1$, $P(x)$ is -2. Since $P(x)$ has changed from $+1$ to -2, there is an irrational zero between $x=0$ and $x=1$. Check the graph to see if the curve crosses the x-axis between 0 and 1. From the table of values, when $x=2$, $P(x)$ is -3; and when $x=3$, $P(x)$ is 4. Again, $P(x)$ has changed from -3 to $+4$. This means that there must be an irrational zero between $x=2$ and $x=3$. Check the graph to see if the curve crosses the x-axis between 2 and 3.

An irrational zero between $x=0$ and $x=1$ means the value of x that makes $P(x)=0$.

Now synthetic division is used to determine the approximate value of the zeros of $P(x)=x^3-2x^2-2x+1$.

The potential rational zeros are ± 1.

Always obtain any rational zeros before attempting to find the approximations for the irrational zeros. The reason for this is that it is much easier to use a polynomial of lesser degree than the original polynomial when using synthetic division.

$$\begin{array}{r|rrrr} +1 & 1 & -2 & -2 & 1 \\ & & 1 & -3 & 1 \\ \hline & 1 & -3 & 1 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x+1$ is a factor and -1 is a zero.

Thus, $P(x)=(x+1)\left(x^2-3x+1\right)$.

The remaining two irrational zeros can be determined by using the quadratic formula to find the zeros of x^2-3x+1. The quadratic formula will not be used so that you can understand the method involved in determining approximations for irrational roots. From the graph, you can see that one irrational zero appears to be between 0 and 0.5. Try 0.3 but use -0.3 as the divisor in the synthetic division method. Remember to use the coefficients of x^2-3x+1.

$$\begin{array}{r|rrr} -0.3 & 1 & -3 & 1 \\ & & -0.3 & 0.81 \\ \hline & 1 & -2.7 & \boxed{0.19} \end{array}$$

Now try the value 0.4 to see if the remainder is closer to zero than it is for 0.3. For the value 0.4, use the divisor -0.4.

$$\begin{array}{r|rrr} -0.4 & 1 & -3 & 1 \\ & & -0.4 & 1.04 \\ \hline & 1 & -2.6 & \boxed{-0.04} \end{array}$$

Since the remainder has changed from a 0.19 to a -0.04, the irrational zero must be between 0.3 and 0.4. Since the remainder -0.04 is closer to zero than the remainder 0.19, the irrational zero is 0.4 (to one decimal place). If the rational zero must be approximated to two decimal places, then divide the interval from 0.3 to 0.4 into ten equal parts. Use synthetic division for the values 0.31, 0.32, 0.33,..., 0.38, 0.39.

Try the value 0.38 since the remainder for 0.4 is closer to zero than the remainder for 0.3. Use the divisor -0.38 in the synthetic division process.

-0.38	1	-3	1
		-0.38	0.9956
	1	-2.62	0.0044

Now try the value 0.39 but use the divisor -0.39.

-0.39	1	-3	1
		-0.39	1.0179
	1	-2.61	-0.0179

Since the remainder for the value 0.38 is closer to zero than the remainder for the value of 0.39, then the irrational zero is 0.38 (to two decimal places).

If the irrational zeros had to be approximated to three decimal places, then divide the interval from 0.38 to 0.39 into ten equal parts. Then synthetic division would be used for the values 0.381, 0.382, 0.383,..., 0.388, 0.389.

The irrational zero between 2 and 3 will now be determined to two decimal places. From the graph, this irrational zero appears to be between 2.5 and 3. Try 2.5, but use -2.5 as the divisor.

-2.5	1	-3	1
		-2.5	1.25
	1	-0.5	-0.25

Now try 2.6, but use the divisor -2.6.

-2.6	1	-3	1
		-2.6	1.04
	1	-0.4	-0.04

Since the remainder is still negative, then try 2.7. Use -2.7 for the divisor.

-2.7	1	-3	1
		-2.7	0.81
	1	-0.3	0.19

Since the remainder for 2.6 is closer to zero than the remainder for 2.7, then the irrational zero is 2.6 (to one decimal place). The irrational zero must be determined to two decimal places. This zero must be between 2.6 and 2.7 since the remainder changes sign from -0.04 to $+0.19$. Divide the interval from 2.6 to 2.7 into ten equal parts. Use synthetic division for the values 2.61, 2.62, 2.63,..., 2.68, 2.69.

Since the remainder for 2.6 is closer to zero than the remainder for 2.7, then try values closer to 2.6. Try 2.61, but use -2.61 for the divisor.

$$\begin{array}{r|rrr} -2.61 & 1 & -3 & 1 \\ & & -2.61 & 1.0179 \\ \hline & 1 & -0.39 & -0.0179 \end{array}$$

Now try 2.62, but use -2.62 for the divisor.

$$\begin{array}{r|rrr} -2.62 & 1 & -3 & 1 \\ & & -2.62 & 0.9956 \\ \hline & 1 & -0.38 & 0.0044 \end{array}$$

Since the remainder for 2.62 is closer to zero than the remainder for 2.61, then the irrational zero is 2.62 (to two decimal places).

Thus, the polynomial has one rational zero, -1, and two irrational zeros, 0.38 and 2.62 (to two decimal places).

The following is a summary of steps for determining approximate irrational zeros of a polynomial.

Step 1: Prepare a table of values for the polynomial; then draw the graph.

Step 2: Determine the potential rational zeros; then use synthetic division to find any rational zeros.

Step 3: Examine the graph to see where the curve crosses or touches the x-axis.

Step 4: Look at the table of values to see where the values for $P(x)$ change from negative to positive or from positive to negative.

Step 5: Use synthetic division to determine which x-value produces the closest remainder to zero.

8. Using the polynomial function $P(x)=x^3-3x^2-2x+2$, prepare a table of values for x-values between -2 and 4 inclusive; then, draw the graph.

9. For the polynomial in question 1, determine all real zeros – rational and irrational. Any irrational zero should be determined to two decimal places. Use synthetic division to determine the irrational zeros. Do not use the quadratic formula.

Check your answers by turning to the Appendix.

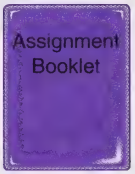
Conclusion

In this section you discovered how to sketch the graph of a given polynomial function. You also worked backwards from a graph to the defining rule of the polynomial function. To assist you in graphing or determining functions, you used the zeros of the polynomials and the multiplicities of those zeros.

NASA

Could an astronaut from the Apollo 17 flight adequately describe the spectacular view of Earth from memory, without using a picture? In mathematics, a graphical representation (a picture) is necessary in order to completely describe a polynomial function. A graph assists in the analysis, helping to classify either what is yet to be found or what has been discovered through algebraic techniques.

Assignment



You are now ready to complete the section assignment.

Module Summary

This module has given you the opportunity to master the vocabulary assocated with and analyse various forms of polynomial functions. You also interpreted the graphs of polynomial functions, exploring the properties of these graphs and how the properties relate to the forms of the functions themselves. As well, you solved equations arising from polynomial functions. You should be aware that both polynomial equations and polynomial functions are useful in describing a variety of real-world situations.

Polynomial functions may be the domain of the mathematician or the serious student. Nevertheless, when skiers compete among themselves, seeing how far they can sail in the air after going over a jump, they are unknowingly applying the fundamentals of this module. Distance, velocity, and acceleration can be related using polynomial functions.

Final Module Assignment

Assignment Booklet

You are now ready to complete the final module assignment.

APPENDIX

Glossary

Diploma Exam Information

Suggested Answers

Glossary

Algorithm: a set of rules for solving a problem in a finite number of steps

Factor Theorem: a theorem which states that a polynomial $P(x)$ is divisible by $(x-a)$ if $P(a)=0$

Integral Zero Theorem: a theorem which states that an integral zero of an integral polynomial $P(x)$ is a factor of its constant term

Multiplicity: the number of times the same root occurs in the solution of an equation

Polynomial equation: a polynomial expression equated to zero

Polynomial function: an expression of the form $f(x)=a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$, where a_0, a_1, ..., a_n are real numbers, $a_n \neq 0$, and $n \in N$

Rational Zero Theorem: a theorem stating that if $\frac{v}{w}$ is a rational zero (in lowest terms) of an integral polynomial $P(x)$, $P(x)=a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$, then v is a factor of a_0 and w is a factor of a_n

Remainder Theorem: a theorem which states that when a polynomial in x is divided by $x-a$, the remainder R is equal to $P(a)$

Synthetic division: a process of dividing a polynomial in one variable by a binomial of the form $x-a$

Zeros of a polynomial: values of x for which $y=0$ for the function $y=f(x)$

Diploma Exam Information

Preparing to Write the Mathematics 30 Diploma Exam – What you should do before the exam!

1. Prepare a course review schedule.

   - Schedule study time for at least the two-week period before the exam.
   - Divide the course material into sections, and indicate on the schedule the time blocks to be devoted to each section.
     - Take into account the exam blueprint available from your teacher (*Diploma Examinations Program Information Bulletin* for the course). This blueprint indicates the relative importance (weighting) of each unit in the course. Note that course units are not equally weighted on the diploma exam.
     - Take into account modules/concepts that you find most difficult. Allocate more time for their review.

2. Obtain and review exam schedules, rules, and policies.

    - Record the time and place of writing.
    - Note minimum and maximum writing times permitted.
    - Prepare to remain in the exam room for 2.5 h (tissues, cough drops, and so on)
    - Identify materials allowed for writing each exam, such as pencils, pens, calculators, mathematical instruments, and a clear plastic ruler.

3. Identify, collect, and practise examples of each type of question that will be asked.

    - Review the format of previous diploma exams.
    - Learn the meanings of the key "directing" words.

4. Use memory aids such as the following:

    - Colour code, underline, highlight, or jot key words in margins.
    - Number points to be memorized.
    - Group word and idea associations.
    - Read aloud key words, expressing key words in your own words.

Suggestions for Students Writing the Mathematics 30 Diploma Exam – What you should know when writing the exam!

1. Make use of the formula sheet at the end of the diploma exam.

2. Do not be afraid to answer each question even if you are not sure of the correct solution to the problem. A penalty is NOT given for guessing on the machine-scored section of the exam. Partial marks are often awarded for incomplete answers in the written-response section of the exam.

3. If you are stuck on a question, mark the alternatives that you know are incorrect and choose from the ones that are left using logical guessing strategy. Think of the questions as challenges, and cultivate a positive attitude about your ability to answer them.

4. Scan the written-response, multiple-choice, and numerical-response sections of the exam before answering the questions. A question in one section of the exam may jog your memory about a question in another section.

5. When first reading a multiple-choice question, locate and circle key words to help clarify the meaning of the question. Then hide the alternatives and try to formulate an answer of your own. This way you will not be distracted by the alternatives.

6. Have a good reason for changing an answer. Do not change an answer on a hunch. Do not waste your time looking for patterns of As, Bs, Cs, or Ds in multiple-choice answers. There are none.

7. You may not have time to write and edit a complete rough copy for each written-response question, but you should prepare an outline of your answer and use it as a guide when writing your good copy.

8. When completing a written-response question, keep in mind the reader of your response. The reader will want to know how well you

   - understand the problem or the mathematical concept
   - can correctly use the mathematics involved
   - can use problem-solving strategies and explain your answer and procedures
   - can communicate your solutions and mathematical ideas

9. Rewriting a statement of the question is often a good way to begin a written response. Conclude with a summary statement. Be sure you have clearly explained all assumptions and have verified your conclusions.

10. Keep track of the time and pace yourself. Put a check mark by items that you are uncertain about and return to them if there is time at the end of the exam.

Suggested Answers

Section 1: Activity 1

1. a. The numerical coefficients are $a_4 = 2$, $a_3 = 0$, $a_2 = 4$, $a_1 = 1$, and $a_0 = -8$.

The degree is 4.
The constant term is -8.
The initial coefficient is 2.

b. The numerical coefficients are $a_5 = 1$, $a_4 = -2$, $a_3 = 0$, $a_2 = 0$, $a_1 = -1$, and $a_0 = 2$.

The degree is 5.
The constant term is 2.
The initial coefficient is 1.

c. The numerical coefficient is $a_0 = 8$.

The degree is 0.
The constant term is 8.
The initial coefficient is 8.

2. a. No, the variable in the denominator has a positive exponent.

b. Yes. (Remember: 8 is the same as $8x^0$.)

c. Yes.

d. No, the middle term does not have a positive integral exponent.

e. Yes. Note that $\frac{5}{x^{-2}}$ is the same as $5x^2$.

f. No, the middle term has a negative exponent.

g. No, the third term $\left(\sqrt{-3}\right)$ is not real.

h. Yes.

i. No, the coefficient of the x^2-term is not real.

j. No, the second term has a variable in the denominator which has a positive integral exponent.

Section 1: Activity 2

1. degree: 2, quadratic
coefficients: integral

2. degree: 4, quartic
coefficients: rational

3. degree: 1, linear
coefficients: integral

4. degree: 0, constant
coefficients: real

5. degree: 4, quartic
coefficients: integral

6. degree: 5, quintic
coefficients: rational

7. degree: 2, quadratic
coefficients: real

8. degree: 2, quadratic
coefficients: integral

9. degree: 5, quintic
coefficients: rational

10. degree: 3, cubic
coefficients: real

Section 1: Activity 3

1. a.

$$
\begin{array}{r}
3x^2 + 7x + 21 \\
x-3\overline{)\,3x^3 - 2x^2 + 0x - 7} \\
\underline{3x^3 - 9x^2} \\
7x^2 + 0x \\
\underline{7x^2 - 21x} \\
21x - 7 \\
\underline{21x - 63} \\
56
\end{array}
$$

b. Arrange the terms in descending powers of x.

$$
\begin{array}{r}
-2x^2 + 8x - 13 \\
x+2\overline{)\,-2x^3 + 4x^2 + 3x - 6} \\
\underline{-2x^3 - 4x^2} \\
8x^2 + 3x \\
\underline{8x^2 + 16x} \\
-13x - 6 \\
\underline{-13x - 26} \\
20
\end{array}
$$

c. Arrange the terms in descending powers of x.

$$
\begin{array}{r}
3x-1 \\
2x-1\overline{)6x^2-5x-8} \\
6x^2-3x \\
\hline
-2x-8 \\
-2x+1 \\
\hline
-9
\end{array}
$$

d.
$$
\begin{array}{r}
4x^3-6x^2+7x-9 \\
2x+3\overline{)8x^4+0x^3-4x^2+3x-5} \\
8x^4+12x^3 \\
\hline
-12x^3-4x^2 \\
-12x^3-18x^2 \\
\hline
14x^2+3x \\
14x^2+21x \\
\hline
-18x-5 \\
-18x-27 \\
\hline
22
\end{array}
$$

2. a.
$$
\begin{array}{r}
4x^2+6x+12 \\
x-2\overline{)4x^3-2x^2+0x-6} \\
4x^3-8x^2 \\
\hline
6x^2+0x \\
6x^2-12x \\
\hline
12x-6 \\
12x-24 \\
\hline
18
\end{array}
$$

$Q(x)=4x^2+6x+12$

$R=18$

b.
$$
\begin{array}{rl}
4x^2+6x+12 & \leftarrow Q(x) \\
x-2 & \leftarrow D(x) \\
\hline
-8x^2-12x-24 & \\
4x^3+6x^2+12x & \\
\hline
4x^3-2x^2+0x-24 & \\
18 & \leftarrow R \\
\hline
4x^3-2x^2-6 & \leftarrow P(x)
\end{array}
$$

Thus, $P(x)=[D(x)\times Q(x)]+R$.

c. Since R does not equal 0, then $D(x)$ is not a factor of $P(x)$.

3. a.

$$x+1\overline{)3x^3+0x^2-6x+4}$$

$$\begin{array}{r} 3x^2-3x-3 \\ \hline 3x^3+3x^2 \\ -3x^2-6x \\ \underline{-3x^2-3x} \\ -3x+4 \\ \underline{-3x-3} \\ 7 \end{array}$$

$Q(x)=3x^2-3x-3$

$R=7$

b.

$$\begin{array}{r} 3x^2-3x-3 \quad \leftarrow Q(x) \\ \hline x+1 \quad \leftarrow D(x) \\ \hline 3x^2-3x-3 \\ \hline 3x^3-3x^2-3x \\ \hline 3x^3+0x^2-6x-3 \\ \hline 7 \quad \leftarrow R \\ \hline 3x^3 \qquad -6x+4 \quad \leftarrow P(x) \end{array}$$

Thus, $P(x)=[D(x)\times Q(x)]+R$.

c. Since R does not equal 0, then $D(x)$ is not a factor of $P(x)$.

4. a.

$$x-1\overline{)2x^4+0x^3+0x^2-5x+3}$$

$$\begin{array}{r} 2x^3+2x^2+2x-3 \\ \hline 2x^4-2x^3 \\ 2x^3+0x^2 \\ \underline{2x^3-2x^2} \\ 2x^2-5x \\ \underline{2x^2-2x} \\ -3x+3 \\ \underline{-3x+3} \\ 0 \end{array}$$

$Q(x)=2x^3+2x^2+2x-3$

$R=0$

b.

$$\begin{array}{r} 2x^3+2x^2+2x-3 \quad \leftarrow Q(x) \\ \hline x-1 \quad \leftarrow D(x) \\ \hline -2x^3-2x^2-2x+3 \\ \hline 2x^4+2x^3+2x^2-3x \\ \hline 2x^4+0x^3+0x^2-5x+3 \\ \hline 0 \quad \leftarrow R \\ \hline 2x^4 \qquad -5x+3 \quad \leftarrow P(x) \end{array}$$

Thus, $P(x)=[D(x)\times Q(x)]+R$.

c. Since the R is zero, $D(x)$ is a factor of $P(x)$.

Section 1: Activity 4

1.

-3	1	0	-2	-4	-6	12
		-3	-9	-21	-51	-135
	1	3	7	17	45	147

$Q(x)=x^4+3x^3+7x^2+17x+45$

$R=147$

2.

2	-5	2	0	-3
		-10	24	-48
	-5	12	-24	45

$Q(x)=-5x^2+12x-24$

$R=45$

3.

$\frac{1}{2}$	4	-6	2	10
		2	-4	3
	4	-8	6	7

$Q(x)=4x^2-8x+6$

$R=7$

4.

3	1	0	-6	0	-27
		3	-9	9	-27
	1	-3	3	-9	0

$Q(x)=x^3-3x^2+3x-9$

$R=0$

5. a.

$-\frac{1}{4}$	4	7	-14	-13	4
		-1	-2	3	4
	4	8	-12	-16	0

$Q(x)=4x^3+8x^2-12x-16$

$R=0$

b. Since the remainder is 0, then $x-\frac{1}{4}$ is a factor.

$$\begin{aligned}
&4x^4+7x^3-14x^2-13x+4\\
&=\left(x-\frac{1}{4}\right)\left(4x^3+8x^2-12x-16\right)\\
&=\left(x-\frac{1}{4}\right)(4)\left(x^3+2x^2-3x-4\right)\\
&=(4x-1)\left(x^3+2x^2-3x-4\right)
\end{aligned}$$

6. a.

$\frac{1}{5}$	5	1	-10	-2	5	1
		1	0	-2	0	1
	5	0	-10	0	5	0

$Q(x)=5x^4-10x^2+5$

$R=0$

b. Since the remainder is 0, then $x+\frac{1}{5}$ is a factor of $P(x)$.

$$\begin{aligned}
&5x^5+x^4-10x^3-2x^2+5x+1\\
&=\left(x+\frac{1}{5}\right)\left(5x^4-10x^2+5\right)\\
&=\left(x+\frac{1}{5}\right)(5)\left(x^4-2x^2+1\right)\\
&=(5x+1)\left(x^4-2x^2+1\right)
\end{aligned}$$

7. a. Change $3x-1$ to $3\left(x-\frac{1}{3}\right)$. Thus, divide by $x-\frac{1}{3}$.

$$\begin{array}{r|cccc} -\frac{1}{3} & 3 & -1 & -6 & -7 & 3 \\ & & -1 & 0 & 2 & 3 \\ \hline & 3 & 0 & -6 & -9 & \boxed{0} \end{array}$$

b. The quotient coefficients 3, 0, -6, and -9 are for the divisor $x-\frac{1}{3}$. To obtain the quotient when the divisor is $3x-1$, divide the coefficients by 3.

$Q(x)=x^3-2x-3$ (when divisor is $3x-1$)
$R=0$

c. Since the remainder is 0, then $x-\frac{1}{3}$ is a factor of $P(x)$.

$$\begin{aligned} 3x^4-1x^3-6x^2-7x+3 &= \left(x-\tfrac{1}{3}\right)\left(3x^3-6x-9\right) \\ &= \left(x-\tfrac{1}{3}\right)(3)\left(x^3-2x-3\right) \\ &= (3x-1)\left(x^3-2x-3\right) \end{aligned}$$

Section 1: Follow-up Activities

Extra Help

1.
$$\begin{array}{r|cccc} 3 & 7 & -2 & 0 & -5 & -3 \\ & & 21 & -69 & 207 & -636 \\ \hline & 7 & -23 & 69 & -212 & \boxed{633} \end{array}$$

$Q(x)=7x^3-23x^2+69x-212$
$R=633$

2.
$$\begin{array}{r|cccc} -1 & 2 & -7 & 4 & 0 \\ & & -2 & 5 & 1 \\ \hline & 2 & -5 & -1 & \boxed{-1} \end{array}$$

$P(x)=2x^2-5x-1$
$R=-1$

3. Change $2x+1$ to $2\left(x+\frac{1}{2}\right)$. Now, divide $P(x)$ by $\left(x+\frac{1}{2}\right)$.

$$\begin{array}{r|cccccc} \frac{1}{2} & 2 & -3 & -2 & 0 & -8 & -4 \\ & & 1 & -2 & 0 & 0 & -4 \\ \hline & 2 & -4 & 0 & 0 & -8 & \boxed{0} \end{array}$$

To obtain the quotient for the divisor $2x+1$, divide these coefficients by 2. Thus, $Q(x)=x^4-2x^3-4$ for the divisor $2x+1$. The remainder is 0.

Thus, $x+\frac{1}{2}$ is a factor of $2x^5-3x^4-2x^3-8x-4$.

$$P(x)=\left(x+\frac{1}{2}\right)\left(2x^4-4x^3-8\right)$$
$$=\left(x+\frac{1}{2}\right)(2)\left(x^4-2x^3-4\right)$$
$$=(2x+1)\left(x^4-2x^3-4\right)$$

4.

4	−1	0	14	−1	4
		−4	16	−8	28
	−1	4	−2	7	−24

$Q(x)=-x^3+4x^2-2x+7$

$R=-24$

Enrichment

1.

3	2	−1	0	3	−8
		6	15	45	144
	2	5	15	48	136

$Q(x)=2x^3+5x^2+15x+48$

$R=136$

$$P(x)=[D(x)\times Q(x)]+R$$
$$2x^4-x^3+3x-8=(x-3)\left(2x^3+5x^2+15x+48\right)+136$$

2.

$-\frac{1}{2}$	8	0	0	−5
		−4	2	−1
	8	−4	2	−6

$Q(x)=8x^2-4x+2$

$R=-6$

$$P(x)=[D(x)\times Q(x)]+R$$
$$8x^3-5=\left(x+\frac{1}{2}\right)\left(8x^2-4x+2\right)+(-6)$$

3. a.

−3	2	7	−1	−7	15
		−6	−3	12	−15
	2	1	−4	5	0

b. $Q(x)=2x^3+x^2-4x+5$

$R=0$

c. Since the remainder is 0, then $x+3$ is a factor of $P(x)$.

$$\therefore\ 2x^4+7x^3-x^2-7x+15$$
$$=(x+3)\left(2x^3+x^2-4x+5\right)$$

4. When $R=0$, $D(x)$ is a factor of $P(x)$.

5. The remainder must have a lower degree than the divisor. Therefore, the possible values are 0 and 1.

6. Degree of $Q(x)=9-5$
$=4$

$R(x)$ must have a degree less than the degree of $D(x)$. Therefore, the highest degree that $R(x)$ may be is 4.

Section 2: Activity 1

1. a. $P(-1)=-(-1)^4+2(-1)^2+3(-1)-8$
$=-1+2-3-8$
$=-10$

b. $P(2) = -(2)^4 + 2(2)^2 + 3(2) - 8$
$= -16 + 8 + 6 - 8$
$= -10$

c. $P(\sqrt{3}) = -(\sqrt{3})^4 + 2(\sqrt{3})^2 + 3(\sqrt{3}) - 8$
$= -9 + 6 + 3\sqrt{3} - 8$
$= -11 + 3\sqrt{3}$

d. $P(0) = -(0)^4 + 2(0)^2 + 3(0) - 8$
$= -8$

e. $P(k) = -(k)^4 + 2(k)^2 + 3(k) - 8$
$= -k^4 + 2k^2 + 3k - 8$

2. a. $P(\sqrt{5}) = 3(\sqrt{5})^2 - 5$
$= 15 - 5$
$= 10$

b. $P\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^2 - 5$
$= \frac{1}{3} - 5$
$= -4\frac{2}{3}$

c. $P(0) = 3(0)^2 - 5$
$= -5$

d. $P(-2) = 3(-2)^2 - 5$
$= 12 - 5$
$= 7$

e. $P(k) - 3k^2 - 5$

3. a. $P\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 - 8$
$= -\frac{1}{4} - \frac{3}{4} - 8$
$= -9$

b. $P(-2) = 2(-2)^3 - 3(-2)^2 - 8$
$= -16 - 12 - 8$
$= -36$

c. $P(3) = 2(3)^3 - 3(3)^2 - 8$
$= 54 - 27 - 8$
$= 19$

d. $P\left(\frac{1}{3}\right) = 2\left(\frac{1}{3}\right)^3 - 3\left(\frac{1}{3}\right)^2 - 8$
$= 2\left(\frac{1}{27}\right) - 3\left(\frac{1}{9}\right) - 8$
$= -8\frac{7}{27}$

e. $P(0) = 2(0)^3 - 3(0)^2 - 8$
$= -8$

f. $P(k) = 2k^3 - 3k^2 - 8$

Section 2: Activity 2

1.

$P(x)$	Divisor	R	
$-3x^2-4x+5$	$x-3$	-34	
$4x^2+x-8$	$x-3$	31	
$-2x^3-3x+7$	$x-3$	-56	
$3x^3-2x^2-3$	$x-3$	60	

The calculations are as follows:

$$\begin{array}{r|rrr} -3 & -3 & -4 & 5 \\ & & 9 & 39 \\ \hline & -3 & -13 & \boxed{-34} \end{array} \qquad \begin{array}{r|rrr} -3 & 4 & 1 & -8 \\ & & -12 & -39 \\ \hline & 4 & 13 & \boxed{31} \end{array}$$

$$\begin{array}{r|rrrr} -3 & -2 & 0 & -3 & 7 \\ & & 6 & 18 & 63 \\ \hline & -2 & -6 & -21 & \boxed{-56} \end{array} \qquad \begin{array}{r|rrrr} -3 & 3 & -2 & 0 & -3 \\ & & -9 & -21 & -63 \\ \hline & 3 & 7 & 21 & \boxed{60} \end{array}$$

2.

$P(x)$	Divisor	R	$P(3)$
$-3x^2-4x+5$	$x-3$	-34	-34
$4x^2+x-8$	$x-3$	31	31
$-2x^3-3x+7$	$x-3$	-56	-56
$3x^3-2x^2-3$	$x-3$	60	60

The calculations are as follows:

$$\begin{aligned} P(3) &= -3(3)^2-4(3)+5 \\ &= -27-12+5 \\ &= -34 \end{aligned} \qquad \begin{aligned} P(3) &= 4(3)^2+3-8 \\ &= 36+3-8 \\ &= 31 \end{aligned}$$

$$\begin{aligned} P(3) &= -2(3)^3-3(3)+7 \\ &= -54-9+7 \\ &= -56 \end{aligned} \qquad \begin{aligned} P(3) &= 3(3)^3-2(3)^2-3 \\ &= 81-18-3 \\ &= 60 \end{aligned}$$

3. The values for R and $P(3)$ are the same for each polynomial. If a polynomial is divided by $x-a$, the remainder obtained is equal to the value of the polynomial when the value of a is substituted for the variables in the polynomial.

4. a. The divisor is $x-2$, and $x-2$ is equivalent to $x-a$.
Therefore, $a=2$.

$$P(2)=-2(2)^3+6(2)-4$$
$$=-16+12-4$$
$$=-8$$

-2	-2	0	6	-4
		4	8	4
	-2	-4	-2	$\boxed{-8}$

b. The divisor is $x+1$, and $x+1$ is equivalent to $x-a$.
Therefore, $a=-1$.

$$P(-1)=2(-1)^3-(-1)-3$$
$$=-2+1-3$$
$$=-4$$

1	2	0	-1	-3
		2	-2	1
	2	-2	1	$\boxed{-4}$

c. The divisor is $x+2$, and $x+2$ is equivalent to $x-a$.
Therefore, $a=-2$.

$$P(-2)=-1(-2)^4+2(-2)^3-(-2)+4$$
$$=-16-16+2+4$$
$$=-26$$

2	-1	2	0	-1	4
		-2	8	-16	30
	-1	4	-8	15	$\boxed{-26}$

5. a. $P(x)=x^5-3x^4+3x-4$

$$P(2)=(2)^5-3(2)^4+3(2)-4$$
$$=32-48+6-4$$
$$=-14$$

b. $P(x)=3x^3-9x^2-3x+4$

$$P(-3)=3(-3)^3-9(-3)^2-3(-3)+4$$
$$=-81-81+9+4$$
$$=-149$$

6. a. The divisor is $x+2$, and $x+2$ is equivalent to $x-a$.
Therefore, $a=-2$.

$$P(-2)=3(-2)^3-2(-2)^2-d(-2)-4$$
$$-34=-24-8+2d-4$$
$$-34=-36+2d$$
$$2d=2$$
$$d=1$$

b. The divisor is $x+1$, and $x+1$ is equivalent to $x-a$.
Therefore, $a=-1$.

$$P(-1)=-4(-1)^4+2(-1)^3+d(-1)^2-3(-1)+9$$
$$4=-4-2+1d+3+9$$
$$4=6+d$$
$$d=-2$$

7. When the divisor is $x+3$, $x+3$ is equivalent to $x-a$. Therefore, $a=-3$.

$$P(-3)=2(-3)^4+d(-3)^3-c(-3)^2+5(-3)-8$$
$$-41=162-27d-9c-15-8$$
$$-41=139-27d-9c$$
$$-27d-9c=-180 \quad (1)$$

When the divisor is $x-2$, $x-2$ is equivalent to $x-a$. Therefore, $a=2$.

$$P(2)=2(2)^4+d(2)^3-c(2)^2+5(2)-8$$
$$74=32+8d-4c+10-8$$
$$74=34+8d-4c$$
$$8d-4c=40 \quad (2)$$

$4\times(1)$: $-108d-36c=-720$ (3)

$-9\times(2)$: $-72d+36c=-360$ (4)

(4)+(3): $-180d=-1080$

$$d=6$$

Substitute $d=6$ in (1).

$$-27(6)-9c=-180$$
$$-162-9c=-180$$
$$-9c=-18$$
$$c=2$$

Section 2: Activity 3

1. If $x-1$ is equivalent to $x-a$, then $a=1$.

$$P(1)=2(1)^4-6(1)^3+8(1)^2-2(1)-2$$
$$=2-6+8-2-2$$
$$=0$$

Since $P(1)=0$, then $x-1$ is a factor of the polynomial.

2. If $x+\frac{1}{3}$ is equivalent to $x-a$, then $a=-\frac{1}{3}$.

$$P\left(-\frac{1}{3}\right)=3\left(-\frac{1}{3}\right)^3-2\left(-\frac{1}{3}\right)^2+5\left(-\frac{1}{3}\right)+2$$
$$=-\frac{1}{9}-\frac{2}{9}-\frac{5}{3}+2$$
$$=0$$

Since $P\left(-\frac{1}{3}\right)=0$, then $x+\frac{1}{3}$ is a factor of the polynomial.

3. If $x+2$ is equivalent to $x-a$, then $a=-2$.

$$P(-2)=-2(-2)^4+(-2)^3+7(-2)^2-3(-2)+6$$
$$=-32-8+28+6+6$$
$$=0$$

Since $P(-2)=0$, then $x+2$ is a factor of the polynomial.

4. If $x+\frac{1}{2}$ is equivalent to $x-a$, then $a=-\frac{1}{2}$.

$$\therefore P\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)^4-7\left(-\frac{1}{2}\right)^3-4\left(-\frac{1}{2}\right)^2+6\left(-\frac{1}{2}\right)+3$$
$$=\frac{1}{8}+\frac{7}{8}-1-3+3$$
$$=0$$

Since $P\left(-\frac{1}{2}\right)=0$, then $x+\frac{1}{2}$ is a factor of the polynomial.

5. If $x+\sqrt{2}$ is equivalent to $x-a$, then $a=-\sqrt{2}$.

$$\therefore P\left(-\sqrt{2}\right)=4\left(-\sqrt{2}\right)^3+3\left(-\sqrt{2}\right)^2-8\left(-\sqrt{2}\right)-6$$
$$=-8\sqrt{2}+6+8\sqrt{2}-6$$
$$=0$$

Since $P\left(-\sqrt{2}\right)=0$, then $x+\sqrt{2}$ is a factor of the polynomial.

Section 2: Follow-up Activities

Extra Help

1. a. When evaluating a polynomial function, a given value is substituted for the **variable**, and a calculated value is obtained for $P(x)$.

b. If a polynomial $P(x)$ is divided by $x-a$, the **remainder** is equal to $P(a)$.

c. A polynomial $P(x)$ has the binomial $x-a$ as a **factor** only if $P(a)=0$.

d. The number a is a **zero** of a polynomial only if $P(a)=0$.

2. a. $P(-2)=-3(-2)^4-2(-2)^2-8$
$=-48-8-8$
$=-64$

b. $P\left(\sqrt{2}\right)=-3\left(\sqrt{2}\right)^4-2\left(\sqrt{2}\right)^2-8$
$=-12-4-8$
$=-24$

3. a. If $x-2$ is equivalent to $x-a$, then $a=2$.

$P(2)=3(2)^4-4(2)^3+5(2)-8$
$=48-32+10-8$
$=18$

-2	3	-4	0	5	-8
		-6	-4	-8	-26
	3	2	4	13	18

b. If $x+4$ is equivalent to $x-a$, then $a=-4$.

$$P(-4)=(-4)^3-4(-4)+9$$
$$=-64+16+9$$
$$=-39$$

4	1	0	−4	9
		4	−16	48
	1	−4	12	−39

4. If $x+2$ is equivalent to $x-a$, then $a=-2$.

$$P(-2)=-(-2)^4-5(-2)^3+4(-2)^2-d(-2)+3$$
$$61=-16+40+16+2d+3$$
$$61=43+2d$$
$$18=2d$$
$$d=9$$

Enrichment

The nested form for questions 1 to 3 is as follows:

$$4x^3-5x^2-2x+4=\left(4x^2-5x-2\right)x+4$$
$$=\left[(4x-5)x-2\right]x+4$$

1. $f(3)=$ ((4 × 3 − 5) × 3 − 2) × 3 + 4 =

61.

2. $f(-9)=$ ((4 × 9 +/− − 5) × 9 +/− − 3) × 9 +/− + 4 =

-3299.

3. $f(6)=$ ((4 × 6 − 5) × 6 − 2) × 6 + 4 =

676.

Section 3: Activity 1

1. a. polynomial
It is an expression containing a sum of terms.

b. polynomial equation
It is an expression containing a sum of terms equated to zero.

c. polynomial
It is an expression containing a sum of terms.

d. polynomial equation
It is an expression containing a sum of terms which can be equated to zero.

2. a. $x^2+5x+6=(-3)^2+5(-3)+6$

$=9-15+6$

$=0$

Yes, $x=-3$ makes the polynomial zero.

b. No, because polynomials do not have roots.

3. a. $P(x)=x^2-9$

$P(-3)=(-3)^2-9$

$=0$

Yes, $x=-3$ makes the polynomial function zero.

b. No, because polynomial functions do not have roots.

4. a. The potential integral zeros are factors of the constant -7.
Thus, the potential integral zeros are ±1 and ±7.

b. The potential integral zeros are factors of the constant -8.
Thus, the potential integral zeros are ±1, ±2, ±4, and ±8.

c. The potential integral zeros are factors of the constant -9.
Thus, the potential integral zeros are ±1, ±3, and ±9.

d. The potential integral zeros are factors of the constant 10.
Thus, the potential integral zeros are ±1, ±2, ±5, and ±10.

e. The potential integral zeros are factors of the constant -12.
Thus, the potential integral zeros are ±1, ±2, ±3, ±4, ±6, and ±12.

5. a. The potential integral zeros are factors of the constant 8.
Thus, the potential integral zeros are ±1, ±2, ±4, and ±8.

$P(1)=2(1)^3+3(1)^2-18(1)+8$

$\neq0$

Thus, 1 is not a zero.

$P(-1)=2(-1)^3+3(-1)^2-18(-1)+8$

$\neq0$

Thus, -1 is not a zero.

$P(2)=2(2)^3+3(2)^2-18(2)+8$

$=16+12-36+8$

$=0$

Thus, 2 is a zero.

If 2 is a zero, then $x-2$ is a factor. Direct substitution can be continued to find any further integral zeros, or synthetic division can be used at this point to find $Q(x)$ when $x-2$ is the divisor.

$$\begin{array}{r|cccc} -2 & 2 & 3 & -18 & 8 \\ & & -4 & -14 & 8 \\ \hline & 2 & 7 & -4 & 0 \end{array}$$

Since the remainder is 0, then $x-2$ is a factor and 2 is a zero. This agrees with the direct substitution method.

Thus, $P(x)=(x-2)\left(2x^2+7x-4\right)$.

$Q(x)$, which is $2x^2+7x-4$, can now be factored to find the remaining zeros.

$$\left(2x^2+7x-4\right)=(2x-1)(x+4)$$

Let $(2x-1)(x+4)=0$.

$2x-1=0$ or $x+4=0$

$2x=1$ $\qquad$ $x=-4$

$x=\frac{1}{2}$

The integral zeros are 2 and -4. The zero $\frac{1}{2}$ is a rational zero, not an integral zero.

b. The potential integral zeros are ±1, ±2, ±3, ±4, ±6, and ±12 (factors of the constant term).

$$P(1)=(1)^3-7(1)^2+16(1)-12$$
$$\neq 0$$

Thus, 1 is not a zero.

$$P(-1)=(-1)^3-7(-1)^2+16(-1)-12$$
$$\neq 0$$

Thus, -1 is not a zero.

$$P(2)=(2)^3-7(2)^2+16(2)-12$$
$$=8-28+32-12$$
$$=0$$

Thus, 2 is a zero.

Synthetic division will now be used to find the remaining zeros. If 2 is a zero, then $x-2$ is a factor. Thus, start with -2 in the divisor.

-2	1	-7	16	-12
		-2	10	-12
	1	-5	6	0

Since the remainder is 0, then $x-2$ is a factor and 2 is a zero. This checks with direct substitution.

$$P(x)=(x-2)\left(1x^2-5x+6\right)$$

$$\left(1x^2-5x+6\right)=(x-2)(x-3)$$

Let $(x-2)(x-3)=0$.

$x-2=0$ or $x-3=0$

$x=2$ $\qquad$ $x=3$

The integral zeros are 2, 2, and 3. This shows that there is a double root at 2. The complete factorization of $P(x)$ is $(x-2)(x-2)(x-3)$. Since a number may be a zero more than once, check $Q(x)$ at each stage of the synthetic division method to see if a root already obtained does indeed repeat.

c. The potential integral zeros are ± 1, ± 2, ± 4, ± 8, ± 16, and ± 32 (factors of the constant term).

$$P(1) = 4(1)^3 + 15(1)^2 - 132(1) + 32$$
$$\neq 0$$

Thus, 1 is not a zero.

$$P(-1) = 4(-1)^3 + 15(-1)^2 - 132(-1) + 32$$
$$\neq 0$$

Thus, -1 is not a zero.

$$P(2) = 4(2)^3 + 15(2)^2 - 132(2) + 32$$
$$= 32 + 60 - 264 + 32$$
$$\neq 0$$

Thus, 2 is not a zero.

$$P(-2) = 4(-2)^3 + 15(-2)^2 - 132(-2) + 32$$
$$= -32 + 60 + 264 + 32$$
$$\neq 0$$

Thus, -2 is not a zero.

$$P(4) = 4(4)^3 + 15(4)^2 - 132(4) + 32$$
$$= 256 + 240 - 528 + 32$$
$$= 0$$

Thus, 4 is a zero.

Synthetic division will now be used to complete the problem. If 4 is a zero, then $x-4$ is a factor. Thus, start with -4 in the divisor.

-4	4	15	-132	32
		-16	-124	32
	4	31	-8	0

Since the remainder is 0, then $x-4$ is a factor and 4 is a zero. This checks with direct substitution.

$$P(x) = (x-4)\left(4x^2 + 31x - 8\right)$$

$$\left(4x^2 + 31x - 8\right) = (4x-1)(x+8)$$

Let $(4x-1)(x+8) = 0$.

$$4x - 1 = 0 \quad \text{or} \quad x + 8 = 0$$
$$4x = 1 \qquad\qquad x = -8$$
$$x = \frac{1}{4}$$

The integral zeros are 4 and -8. The zero $\frac{1}{4}$ is a rational zero, not an integral zero.

d. The potential integral zeros are $\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 20$, and ± 40 (factors of the constant term).

$$P(1) = (1)^4 + 7(1)^3 + 6(1)^2 - 28(1) - 40$$
$$\neq 0$$

Thus, 1 is not a zero.

$$P(-1) = (-1)^4 + 7(-1)^3 + 6(-1)^2 - 28(-1) - 40$$
$$\neq 0$$

Thus, -1 is not a zero.

$$P(2) = (2)^4 + 7(2)^3 + 6(2)^2 - 28(2) - 40$$
$$= 16 + 56 + 24 - 56 - 40$$
$$= 0$$

Thus, 2 is a zero.

Synthetic division will now be used to complete the problem. If 2 is a zero, then $x-2$ is a factor. Start with -2 in the divisor.

-2	1	7	6	-28	-40
		-2	-18	-48	-40
	1	9	24	20	0

Since the remainder is 0, then $x-2$ is a factor and 2 is a zero. This checks with direct substitution.

$$P(x) = (x-2)\left(x^3 + 9x^2 + 24x + 20\right)$$

-2	1	9	24	20
		-2	-22	-92
	1	11	46	112

Since the remainder is not 0, then 2 does not repeat as a zero.

2	1	9	24	20
		2	14	20
	1	7	10	0

Since the remainder is 0, then $x+2$ is a factor and -2 is a zero.

Thus, $P(x) = (x-2)(x+2)\left(x^2 + 7x + 10\right)$.

$$\left(x^2 + 7x + 10\right) = (x+2)(x+5)$$

Let $(x+2)(x+5) = 0$.

$(x+2) = 0$ or $x+5 = 0$

$x = -2$ $\qquad$ $x = -5$

The integral zeros are $2, -2, -2$, and -5. Note that -2 does repeat.

Section 3: Activity 2

1. a. The potential numerators are factors of the constant 8. Thus, the potential numerators are ± 1, ± 2, ± 4, and ± 8.

The potential denominators are factors of the initial coefficient 2. Thus, the potential denominators are ± 1 and ± 2.

The potential rational zeros are ± 1, ± 2, ± 4, ± 8, and $\pm \frac{1}{2}$.

b. The potential numerators are ± 1, ± 3, and ± 9 (factors of the constant term).

The potential denominators are ± 1 and ± 3 (factors of the intial coefficient).

The potential rational zeros are ± 1, ± 3, ± 9, and $\pm \frac{1}{3}$.

c. The potential numerators are ± 1, ± 2, ± 3, and ± 6 (factors of the constant term).

The potential denominators are ± 1 and ± 5 (factors of the initial coefficient).

The potential rational zeros are ± 1, ± 2, ± 3, ± 6, $\pm \frac{1}{5}$, $\pm \frac{2}{5}$, $\pm \frac{3}{5}$, and $\pm \frac{6}{5}$.

d. The potential numerators are ± 1, ± 2, ± 5, and ± 10 (factors of the constant term).

The potential denominators are ± 1, ± 2, and ± 4 (factors of the initial coefficient).

The potential rational zeros are ± 1, ± 2, ± 5, ± 10, $\pm \frac{1}{2}$, $\pm \frac{5}{2}$, $\pm \frac{1}{4}$, and $\pm \frac{5}{4}$.

e. The potential numerators are ± 1, ± 3, ± 5, and ± 15 (factors of the constant term).

The potential denominators are ± 1, ± 2, ± 4, and ± 8 (factors of the initial coefficient).

The potential rational zeros are ± 1, ± 3, ± 5, ± 15, $\pm \frac{1}{2}$, $\pm \frac{3}{2}$, $\pm \frac{5}{2}$, $\pm \frac{15}{2}$, $\pm \frac{1}{4}$, $\pm \frac{3}{4}$, $\pm \frac{5}{4}$, $\pm \frac{15}{4}$, $\pm \frac{1}{8}$, $\pm \frac{3}{8}$, $\pm \frac{5}{8}$, and $\pm \frac{15}{8}$.

2. a. The potential numerators are ± 1 (factors of the constant term).

The potential denominators are ± 1, ± 2, ± 4, and ± 8 (factors of the initial coefficient).

The potential rational zeros are ± 1, $\pm \frac{1}{2}$, $\pm \frac{1}{4}$, and $\pm \frac{1}{8}$.

$$\begin{aligned} P(1) &= 8(1)^3 - 10(1)^2 + 1 + 1 \\ &= 8 - 10 + 1 + 1 \\ &= 0 \end{aligned}$$

Thus, 1 is a zero.

Synthetic division will be used to complete the problem. If 1 is a zero, then $x-1$ is a factor. Start with -1 in the synthetic division method.

$$\begin{array}{r|rrrr} -1 & 8 & -10 & 1 & 1 \\ & & -8 & 2 & 1 \\ \hline & 8 & -2 & -1 & 0 \end{array}$$

Since the remainder is 0, then $x-1$ is a factor and 1 is a zero. This checks with direct substitution.

$$P(x)=(x-1)\left(8x^2-2x-1\right)$$

$$8x^2-2x-1=(4x+1)(2x-1)$$

Let $(4x+1)(2x-1)=0$.

$$\begin{array}{lll} 4x+1=0 & \text{or} & 2x-1=0 \\ 4x=-1 & & 2x=1 \\ x=-\frac{1}{4} & & x=\frac{1}{2} \end{array}$$

The rational zeros are 1, $-\frac{1}{4}$, and $\frac{1}{2}$.

b. The potential numerators are ±1 and ±2.

The potential denominators are ±1, ±2, ±3, ±6, ±9, and ±18.

The potential zeros are ±1, ±2, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, $\pm\frac{1}{6}$, $\pm\frac{1}{9}$, $\pm\frac{2}{9}$, and $\pm\frac{1}{18}$.

$$\begin{aligned} P(1)&=18(1)^3-15(1)^2-1+2 \\ &\neq 0 \end{aligned}$$

Thus, 1 is not a zero.

$$\begin{aligned} P(-1)&=18(-1)^3-15(-1)^2-(-1)+2 \\ &\neq 0 \end{aligned}$$

Thus, -1 is not a zero.

$$\begin{aligned} P(2)&=18(2)^3-15(2)^2-2+2 \\ &=144-60-2+2 \\ &\neq 0 \end{aligned}$$

Thus, 2 is not a zero.

$$\begin{aligned} P(-2)&=18(-2)^3-15(-2)^2-(-2)+2 \\ &=-144-60+2+2 \\ &\neq 0 \end{aligned}$$

Thus, -2 is not a zero.

$$P\left(\frac{1}{2}\right)=18\left(\frac{1}{2}\right)^3-15\left(\frac{1}{2}\right)^2-\frac{1}{2}+2$$
$$=\frac{18}{8}-\frac{15}{4}-\frac{1}{2}+2$$
$$=\frac{18-30-4}{8}+2$$
$$=-2+2$$
$$=0$$

Thus, $\frac{1}{2}$ is a zero.

If $\frac{1}{2}$ is a zero, then $x-\frac{1}{2}$ is a factor. Start with $-\frac{1}{2}$ in the divisor of the synthetic division method.

$$\begin{array}{r|cccc} -\frac{1}{2} & 18 & -15 & -1 & 2 \\ & & -9 & 3 & 2 \\ \hline & 18 & -6 & -4 & 0 \end{array}$$

Since the remainder is 0, then $x-\frac{1}{2}$ is a factor and $\frac{1}{2}$ is a zero. This checks with direct substitution.

$$P(x)=\left(x-\frac{1}{2}\right)\left(18x^2-6x-4\right)$$

$$18x^2-6x-4=2\left(9x^2-3x-2\right)$$
$$=2(3x+1)(3x-2)$$

Let $2(3x+1)(3x-2)=0$.

$3x+1=0$ or $3x-2=0$

$3x=-1$ $\qquad$ $3x=2$

$x=-\frac{1}{3}$ $\qquad$ $x=\frac{2}{3}$

The rational zeros are $\frac{1}{2}$, $-\frac{1}{3}$, and $\frac{2}{3}$.

c. The potential numerators are ±1 and ±2.

The potential denominators are ±1, ±2, ±4, and ±8.

The potential rational zeros are ±1, ±2, $\pm\frac{1}{2}$, $\pm\frac{1}{4}$, and $\pm\frac{1}{8}$.

$$P(1)=8(1)^4-6(1)^3-19(1)^2-3(1)+2$$
$$=8-6-19-3+2$$
$$\neq0$$

Thus, 1 is not a zero.

$$P(-1)=8(-1)^4-6(-1)^3-19(-1)^2-3(-1)+2$$
$$=8+6-19+3+2$$
$$=0$$

Thus, -1 is a zero.

If -1 is a zero, then $x+1$ is a factor. Start with $+1$ in the divisor of the synthetic division method.

$$\begin{array}{r|rrrrr} +1 & 8 & -6 & -19 & -3 & 2 \\ & & 8 & -14 & -5 & 2 \\ \hline & 8 & -14 & -5 & 2 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x+1$ is a factor and -1 is a zero. This checks with direct substitution.

$$P(x)=(x+1)\left(8x^3-14x^2-5x+2\right)$$

Since -1 may be a repeated zero, start the synthetic division by checking -1 again.

$$\begin{array}{r|rrrr} +1 & 8 & -14 & -5 & 2 \\ & & 8 & -22 & 17 \\ \hline & 8 & -22 & 17 & \boxed{-15} \end{array}$$

Since the remainder is -15, the root -1 will not repeat.

$$\begin{array}{r|rrrr} 2 & 8 & -14 & -5 & 2 \\ & & 16 & -60 & 110 \\ \hline & 8 & -30 & 55 & \boxed{-108} \end{array}$$

Since the remainder is not 0, then $x+2$ is not a factor and -2 is not a zero.

$$\begin{array}{r|rrrr} -2 & 8 & -14 & -5 & 2 \\ & & -16 & -4 & 2 \\ \hline & 8 & 2 & -1 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x-2$ is a factor and 2 is a zero.

$$P(x)=(x+1)(x-2)\left(8x^2+2x-1\right)$$

$$8x^2+2x-1=(4x-1)(2x+1)$$

Let $(4x-1)(2x+1)=0$.

$$\begin{array}{lll} 4x-1=0 & \text{or} & 2x+1=0 \\ 4x=1 & & 2x=-1 \\ x=\frac{1}{4} & & x=-\frac{1}{2} \end{array}$$

The rational zeros are -1, 2, $\frac{1}{4}$, and $-\frac{1}{2}$.

d. The potential numerators are ± 1.

The potential denominators are ± 1, ± 2, and ± 4.

The potential rational zeros are ± 1, $\pm\frac{1}{2}$, and $\pm\frac{1}{4}$.

$$P(1)=4(1)^3+7(1)^2+2(1)-1$$
$$\neq 0$$

Thus, 1 is not a zero.

$$P(-1)=4(-1)^3+7(-1)^2+2(-1)-1$$
$$=-4+7-2-1$$
$$=0$$

Thus, -1 is a zero.

If -1 is a zero, then $x+1$ is a factor. Start with $+1$ in the divisor of the synthetic division method.

$+1$	4	7	2	-1
		4	3	-1
	4	3	-1	0

Since the remainder is 0, then $x+1$ is a factor and -1 is a zero. This checks with direct substitution.

$$P(x)=(x+1)\left(4x^2+3x-1\right)$$

$$4x^2+3x-1=(4x-1)(x+1)$$

Let $(4x-1)(x+1)=0$.

$$4x-1=0 \quad \text{or} \quad x+1=0$$
$$4x=1 \qquad\qquad x=-1$$
$$x=\frac{1}{4}$$

The rational zeros are $\frac{1}{4}$ and -1 (multiplicity of two).

Section 3: Activity 3

1. The potential numerators are ±1, ±2, ±3, and ±6 (factors of the constant term).

The potential denominators are ±1 and ±2 (factors of the initial coefficient).

The potential rational zeros are ±1, ±2, ±3, ±6, $\pm\frac{1}{2}$, and $\pm\frac{3}{2}$.

$$P(1)=2(1)^3-3(1)^2-11(1)+6$$
$$\neq 0$$

Thus, 1 is not a zero.

$$P(-1)=2(-1)^3-3(-1)^2-11(-1)+6$$
$$\neq 0$$

Thus, -1 is not a zero.

$$P(2)=2(2)^3-3(2)^2-11(2)+6$$
$$=16-12-22+6$$
$$\neq 0$$

Thus, 2 is not a zero.

$$P(-2)=2(-2)^3-3(-2)^2-11(-2)+6$$
$$=-16-12+22+6$$
$$=0$$

Thus, -2 is a zero.

If -2 is a zero, then $x+2$ is a factor. For the synthetic division method, start the divisor with 2.

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -11 & 6 \\ & & 4 & -14 & 6 \\ \hline & 2 & -7 & 3 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x+2$ is a factor and the zero is -2. This checks with the direct substitution method.

$$\begin{aligned} P(x) &= (x+2)\left(2x^2-7x+3\right) \\ &= (x+2)(2x-1)(x-3) \end{aligned}$$

Let $(2x-1)(x-3)=0$.

$2x-1=0$ or $x-3=0$

$2x=1$ $\qquad$ $x=3$

$x=\frac{1}{2}$

The factored form of $P(x)$ over the integers is $P(x)=(x+2)(2x-1)(x-3)$.

The real zeros of $P(x)$ are -2, $\frac{1}{2}$, and 3.

2. The potential numerators are ±1, ±2, ±4, and ±8.

The potential denominators are ±1.

The potential rational zeros are ±1, ±2, ±4, and ±8.

$$\begin{aligned} P(1) &= (1)^4-3(1)^3-2(1)^2+12(1)-8 \\ &= 1-3-2+12-8 \\ &= 0 \end{aligned}$$

Thus, 1 is a zero and $x-1$ is a factor.

Start with -1 in the divisor of the synthetic division process.

$$\begin{array}{r|rrrr} -1 & 1 & -3 & -2 & 12 & -8 \\ & & -1 & 2 & 4 & -8 \\ \hline & 1 & -2 & -4 & 8 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x-1$ is a factor and 1 is a zero. This checks with direct substitution.

$$P(x)=(x-1)\left(x^3-2x^2-4x+8\right)$$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -4 & 8 \\ & & 1 & -3 & -1 \\ \hline & 1 & -3 & -1 & \boxed{9} \end{array}$$

Therefore, -1 is not a zero.

$$\begin{array}{r|rrrr} 2 & 1 & -2 & -4 & 8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

Since the remainder is zero, $x+2$ is a factor and -2 is a zero.

$P(x)=(x-1)(x+2)\left(x^2-4x+4\right)$

$\left(x^2-4x+4\right)=(x-2)(x-2)$

Let $(x-2)(x-2)=0$.

$x-2=0$ or $x-2=0$

$x=2$ $\quad$ $x=2$

The factored form of $P(x)$ over the integers is

$P(x)=(x-1)(x+2)(x-2)^2$.

The real zeros of $P(x)$ are 1, -2, and 2 (multiplicity of two).

3. The potential numerators are ± 1 (factors of the constant term).

The potential denominators are ± 1, ± 2, ± 3, and ± 6.

The potential rational zeros are ± 1, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, and $\pm\frac{1}{6}$.

$P(1)=6(1)^3+7(1)^2-1$

$\neq 0$

Thus, 1 is not a zero.

$P(-1)=6(-1)^3+7(-1)^2-1$

$=-6+7-1$

$=0$

Thus, -1 is a zero and $x+1$ is a factor.

Start the synthetic division with a divisor of $+1$.

$$\begin{array}{r|cccc} +1 & 6 & 7 & 0 & -1 \\ & & 6 & 1 & -1 \\ \hline & 6 & 1 & -1 & 0 \end{array}$$

Since the remainder is 0, then $x+1$ is a factor and -1 is a zero. This agrees with direct substitution.

$P(x)=(x+1)\left(6x^2+1x-1\right)$

$\left(6x^2+1x-1\right)=(3x-1)(2x+1)$

Let $(3x-1)(2x+1)=0$.

$3x-1=0$ or $2x+1=0$

$3x=1$ $\quad$ $2x=-1$

$x=\frac{1}{3}$ $\quad$ $x=-\frac{1}{2}$

The factored form of $P(x)$ over the integers is

$P(x)=(x+1)(3x-1)(2x+1)$.

The real zeros of $P(x)$ are -1, $\frac{1}{3}$, and $-\frac{1}{2}$.

Section 3: Activity 4

1. a. The potential numerators are ± 1 and ± 2.

The potential denominators are ± 1, ± 2, ± 3, and ± 6.

The potential rational zeros are ± 1, ± 2, $\pm \frac{1}{2}$, $\pm \frac{1}{3}$, and $\pm \frac{2}{3}$.

$$P(1) = 6(1)^3 + 7(1)^2 - 9(1) + 2$$
$$\neq 0$$

Thus, 1 is not a zero.

$$P(-1) = 6(-1)^3 + 7(-1)^2 - 9(-1) + 2$$
$$\neq 0$$

Thus, -1 is not a zero.

$$P(2) = 6(2)^3 + 7(2)^2 - 9(2) + 2$$
$$\neq 0$$

Thus, 2 is not a zero.

$$P(-2) = 6(-2)^3 + 7(-2)^2 - 9(-2) + 2$$
$$= -48 + 28 + 18 + 2$$
$$= 0$$

Thus, -2 is a zero.

Since -2 is a zero, $x+2$ is a factor. Start with $+2$ in the divisor of the synthetic division method.

$$\begin{array}{r|cccc} +2 & 6 & 7 & -9 & 2 \\ & & 12 & -10 & 2 \\ \hline & 6 & -5 & 1 & 0 \end{array}$$

Since the remainder is 0, then $x+2$ is a factor and -2 is a zero. This checks with direct substitution.

The factored form is $(x+2)\left(6x^2 - 5x + 1\right) = 0$. The remaining zeros are zeros of the quotient polynomial $6x^2 - 5x + 1$. To obtain the final zeros, factor the quadratic polynomial or use the quadratic formula.

$$6x^2 - 5x + 1 = (3x - 1)(2x - 1)$$

Let $(3x-1)(2x-1) = 0$.

$$3x - 1 = 0 \quad \text{or} \quad 2x - 1 = 0$$
$$3x = 1 \qquad\qquad 2x = 1$$
$$x = \frac{1}{3} \qquad\qquad x = \frac{1}{2}$$

The roots are -2, $\frac{1}{3}$, and $\frac{1}{2}$.

b. The potential numerators are ±1, ±2, ±3, ±6, ±9, and ±18 (factors of the constant term).

The potential denominators are ±1, ±2, ±4, and ±8.

The potential rational zeros are ±1, ±2, ±3, ±6, ±9, ±18, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{9}{2}$, $\pm\frac{1}{4}$, $\pm\frac{3}{4}$, $\pm\frac{9}{4}$, $\pm\frac{1}{8}$, $\pm\frac{3}{8}$, and $\pm\frac{9}{8}$.

$$P(1)=8(1)^4-6(1)^3-53(1)^2-9(1)+18$$
$$\neq 0$$

Thus, 1 is not a zero.

$$P(-1)=8(-1)^4-6(-1)^3-53(-1)^2-9(-1)+18$$
$$\neq 0$$

Thus, -1 is not a zero.

$$P(2)=8(2)^4-6(2)^3-53(2)^2-9(2)+18$$
$$=128-48-212-18+18$$
$$\neq 0$$

Thus, 2 is not a zero.

$$P(-2)=8(-2)^4-6(-2)^3-53(-2)^2-9(-2)+18$$
$$=128+48-212+18+18$$
$$=0$$

Thus, -2 is a zero.

Since -2 is a zero, $x+2$ is a factor. Start with $+2$ in the divisor of the synthetic division method.

$+2$	8	-6	-53	-9	18
		16	-44	-18	18
	8	-22	-9	9	0

Since the remainder is 0, then $x+2$ is a factor and -2 is a zero. This checks with direct substitution.

The factored form is $(x+2)\left(8x^3-22x^2-9x+9\right)=0$.

Since the quotient polynomial $8x^3-22x^2-9x+9$ is of degree 3, continue using synthetic division on the quotient polynomial.

3	8	-22	-9	9
		24	-138	387
	8	-46	129	-378

The remainder is not zero; thus, -3 is not a root.

-3	8	-22	-9	9
		-24	-6	9
	8	2	-3	0

The remainder is 0; thus, $x-3$ is a factor and 3 is a root.

The factored form is $(x+2)(x-3)\left(8x^2+2x-3\right)=0$.

The remaining zeros will be determined from the polynomial $8x^2+2x-3$.

$$8x^2+2x-3=(4x+3)(2x-1)$$

Let $(4x+3)(2x-1)=0$.

$4x+3=0$ or $2x-1=0$

$4x=-3$ $\quad$ $2x=1$

$x=-\frac{3}{4}$ $\quad$ $x=\frac{1}{2}$

The roots are -2, 3, $-\frac{3}{4}$, and $\frac{1}{2}$.

c. The potential numerators are ±1, ±2, ±3, ±6, ±9, and ±18 (factors of the constant term).

The potential denominators are ±1 (factors of the initial coefficient).

The potential rational zeros are ±1, ±2, ±3, ±6, ±9, and ±18.

$$P(1)=(1)^4-3(1)^3-7(1)^2+15(1)+18$$
$$=1-3-7+15+18$$
$$\neq 0$$

Thus, 1 is not a zero.

$$P(-1)=(-1)^4-3(-1)^3-7(-1)^2+15(-1)+18$$
$$=1+3-7-15+18$$
$$=0$$

Thus, -1 is a zero.

Since -1 is a zero, $x+1$ is a factor. Start with $+1$ in the divisor of the synthetic division method.

$+1$	1	-3	-7	15	18
		1	-4	-3	18
	1	-4	-3	18	0

Since the remainder is 0, then $x+1$ is a factor and -1 is a zero. This checks with direct substitution.

The factored form is $(x+1)\left(x^3-4x^2-3x+8\right)=0$.

Determine the remaining zeros from x^3-4x^2-3x+8.

$+2$	1	-4	-3	18
		2	-12	18
	1	-6	9	0

Since the remainder is 0, then $x+2$ is a factor and -2 is a zero. The factored form is $(x+1)(x+2)\left(x^2-6x+9\right)=0$.

The remaining zeros will be determined from x^2-6x+9.

$$x^2-6x+9=(x-3)(x-3)$$

Let $(x-3)(x-3)=0$.

$x-3=0$ or $x-3=0$

$x=3$ $\quad$ $x=3$

The roots are -1, -2, and 3 (multiplicity of two).

d. The potential numerators are ± 1, ± 2, ± 4, and ± 8 (factors of the constant).

The potential denominators are ± 1 and ± 2 (factors of the initial coefficient).

The potential rational zeros are ± 1, ± 2, ± 4, ± 8, and $\pm \frac{1}{2}$.

$$P(1) = 2(1)^4 + 9(1)^3 + 4(1)^2 - 16(1) - 8$$
$$\neq 0$$

Thus, 1 is not a zero.

$$P(-1) = 2(-1)^4 + 9(-1)^3 + 4(-1)^2 - 16(-1) - 8$$
$$= 2 - 9 + 4 + 16 - 8$$
$$\neq 0$$

Thus, -1 is not a zero.

$$P(2) = 2(2)^4 + 9(2)^3 + 4(2)^2 - 16(2) - 8$$
$$= 32 + 72 + 16 - 32 - 8$$
$$\neq 0$$

Thus, 2 is not a zero.

$$P(-2) = 2(-2)^4 + 9(-2)^3 + 4(-2)^2 - 16(-2) - 8$$
$$= 32 - 72 + 16 + 32 - 8$$
$$= 0$$

Thus, -2 is a zero.

Since -2 is a zero, $x+2$ is a factor. Start with $+2$ in the divisor of the synthetic division method.

$+2$	2	9	4	-16	-8
		4	10	-12	-8
	2	5	-6	-4	0

Since the remainder is 0, then $x+2$ is a factor and -2 is a zero. This checks with direct substitution.

The factored form is $(x+2)\left(2x^3 + 5x^2 - 6x - 4\right) = 0$.

Determine the remaining zeros from $2x^3 + 5x^2 - 6x - 4$.

$+4$	2	5	-6	-4
		8	-12	24
	2	-3	6	-28

Since the remainder is not 0, then -4 is not a zero.

-4	2	5	-6	-4
		-8	-52	-184
	2	13	46	180

Since the remainder is not 0, then 4 is not a zero.

$$\begin{array}{r|cccc} +8 & 2 & 5 & -6 & -4 \\ & & 16 & -88 & 656 \\ \hline & 2 & -11 & 82 & -660 \end{array}$$

Since the remainder is not 0, then -8 is not a zero.

$$\begin{array}{r|cccc} -8 & 2 & 5 & -6 & -4 \\ & & -16 & -168 & -1296 \\ \hline & 2 & 21 & 162 & 1292 \end{array}$$

Since the remainder is not 0, then 8 is not a zero.

$$\begin{array}{r|cccc} +\frac{1}{2} & 2 & 5 & -6 & -4 \\ & & 1 & 2 & -4 \\ \hline & 2 & 4 & -8 & 0 \end{array}$$

Since the remainder is 0, $x+\frac{1}{2}$ is a factor and $-\frac{1}{2}$ is a zero.

The factored form is $(x+2)\left(x+\frac{1}{2}\right)\left(2x^2+4x-8\right)=0$.
This can be changed to
$(x+2)\left(x+\frac{1}{2}\right)(2)\left(x^2+2x-4\right)=0$, which is equivalent to $(x+2)(2x+1)\left(x^2+2x-4\right)=0$. The remaining roots will be determined from the quadratic x^2+2x-4. Since this polynomial cannot be factored by ordinary factoring methods, the quadratic formula must be used.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad \text{(where } a=1,\ b=2, \text{ and } c=-4\text{)}$$

$$=\frac{-2\pm\sqrt{2^2-4(1)(-4)}}{2(1)}$$

$$=\frac{-2\pm\sqrt{4+16}}{2}$$

$$=\frac{-2\pm\sqrt{20}}{2}$$

$$=\frac{-2\pm2\sqrt{5}}{2}$$

$$=-1\pm\sqrt{5}$$

The roots are -2, $-\frac{1}{2}$, $-1+\sqrt{5}$, and $-1-\sqrt{5}$.

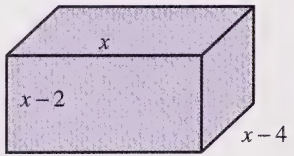
2. Let x be the length.
Let $x-4$ be the width.
Let $x-2$ be the height.

The volume of the freezer compartment is given by the expression $V=x(x-4)(x-2)$. Since $V=48\ \text{cm}^3$, solve for x.

$$x(x-4)(x-2)=48$$

$$x\left(x^2-6x-8\right)=48$$

$$x^3-6x^2+8x-48=0$$



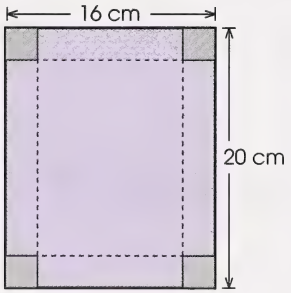
Solve using synthetic division.

$$\begin{array}{r|rrr} -6 & 1 & -6 & 8 & -48 \\ & & -6 & 0 & -48 \\ \hline & 1 & 0 & 8 & \boxed{0} \end{array}$$

Since 6 is a zero, $(x-6)$ is a factor of $x^3-6x^2+8x-48$. The quotient, x^2+8 cannot be factored. Therefore, the length of the freezer compartment is 6 m.

3. Let x represent the side of the square. The volume of the box is expressed as $V=x(16-2x)(20-2x)$. Since $V=384\ \text{cm}^3$, solve for x.

$$\begin{aligned} x(16-2x)(20-2x)&=384 \\ x(2)(8-x)(2)(10-x)&=384 \\ 4x(8-x)(10-x)&=384 \\ x\left(80-18x+x^2\right)&=96 \\ x^3-18x^2+80x-96&=0 \end{aligned}$$



Use synthetic division to find the zeros.

$$\begin{array}{r|rrrr} -2 & 1 & -18 & 80 & -96 \\ & & -2 & 32 & -96 \\ \hline & 1 & -16 & 48 & \boxed{0} \end{array}$$

Since 2 is a zero, then $x-2$ is a factor of $x^3-18x^2+80x-96$ and $Q(x)=x^2-16x+48$.

$$x^2-16x+48=(x-4)(x-12)$$

Since $x-4$ is a factor of $Q(x)$, then 4 is a zero; and since $x-12$ is a factor of $Q(x)$, then 12 is a zero.

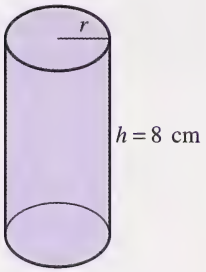
$$x=2,\ 6,\ \text{or}\ 12$$

It is impossible to have the side of a square the length equal to 12 cm. Therefore, the side of a square measures either 2 cm or 6 cm.

4. Area of top and base $=2\pi r^2$
Area of side $=2\pi rh$

The total area is expressed as $A=2\pi r+2\pi rh$.

Since $A=301.44\ \text{cm}^2$, solve for r.



$$2\pi r + 2\pi rh = 301.44$$

$$2(3.14)r^2 + 2(3.14)(8)(r) - 301.44 = 0$$

$$6.28r^2 + 6.28(8r) - 301.44 = 0$$

$$r^2 + 8r - 48 = 0$$

$$(r-4)(r+12) = 0$$

$r - 4 = 0$ or $r + 12 = 0$

$r = 4$ $\qquad$ $r = -12$ (impossible)

The radius is 4 cm.

Volume for a cylinder is $\pi r^2 h$.

$\therefore V = (\pi)(4)^2(8)$

$= 401.92$

The volume is $401.92\ \text{cm}^3$.

Section 3: Follow-up Activities

Extra Help

1. An integral polynomial only has coefficients that are **integers**.

2. The Factor Theorem states that $x - a$ is a factor of $P(x)$ only if $\boldsymbol{P(a) = 0}$.

3. If an integer is a zero of an integral polynomial, then this same integer is a **factor** of the **constant** term of $P(x)$.

4. In synthetic division, a zero of the polynomial is determined when the remainder is zero. If the divisor is -3 when the remainder is 0, then $\boldsymbol{x - 3}$ is a factor and **3** is a zero. If the divisor is 4 when the remainder is 0, then $\boldsymbol{x + 4}$ is a factor and $\boldsymbol{-4}$ is a zero.

5. From the Rational Zero Theorem, the potential numerators are factors of the **constant** term $\boldsymbol{a_0}$. The potential denominators are factors of the coefficient $\boldsymbol{a_n}$ of the **highest** degree term.

6. A polynomial **equation** is a polynomial expression equated to zero.

7. The **multiplicity** of a root is the number of times the same root occurs in the solution.

8. Potential integral zeros are factors of the constant term. Thus, the potential integral zeros are ± 1, ± 2, ± 3, ± 6, ± 9, and ± 18.

9. The potential integral zeros are ± 1, ± 2, ± 3, and ± 6.

1	2	−3	−11	6
		2	−5	−6
	2	−5	−6	12

Since the remainder is not 0, then $x + 1$ is not a factor and -1 is not a zero.

$$\begin{array}{r|rrrr} -1 & 2 & -3 & -11 & 6 \\ & & -2 & 1 & 12 \\ \hline & 2 & -1 & -12 & \boxed{-6} \end{array}$$

Since the remainder is not 0, then $x-1$ is not a factor and 1 is not a zero.

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -11 & 6 \\ & & 4 & -14 & 6 \\ \hline & 2 & -7 & 3 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x+2$ is a factor and -2 is a zero.

$$\begin{aligned} P(x) &= (x+2)\left(2x^2-7x+3\right) \\ &= (x+2)(2x-1)(x-3) \end{aligned}$$

Let $(2x-1)(x-3)=0$.

$$\begin{array}{lll} 2x-1=0 & \text{or} & x-3=0 \\ 2x=1 & & x=3 \\ x=\frac{1}{2} & & \end{array}$$

The integral zeros are -2 and 3.

10. The potential numerators are ±1 and ±7 (factors of the constant term).

The potential denominators are ±1 and ±3.

The potential rational zeros are ±1, ±7, $\pm\frac{1}{3}$, and $\pm\frac{7}{3}$.

11. The potential numerators are ±1 and ±5.

The potential denominators are ±1, ±2, and ±4.

The potential rational zeros are ±1, ±5, $\pm\frac{1}{2}$, $\pm\frac{5}{2}$, $\pm\frac{1}{4}$, and $\pm\frac{5}{4}$.

$$\begin{array}{r|rrrr} 1 & 4 & 17 & -16 & -5 \\ & & 4 & 13 & -29 \\ \hline & 4 & 13 & -29 & \boxed{24} \end{array}$$

Since the remainder is not 0, then $x+1$ is not a factor and -1 is not a zero.

$$\begin{array}{r|rrrr} -1 & 4 & 17 & -16 & -5 \\ & & -4 & -21 & -5 \\ \hline & 4 & 21 & 5 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x-1$ is a factor and 1 is a zero.

$$\begin{aligned} P(x) &= (x-1)\left(4x^2+21x+5\right) \\ &= (x-1)(4x+1)(x+5) \end{aligned}$$

Let $(4x+1)(x+5)=0$.

$$\begin{array}{lll} 4x+1=0 & \text{or} & x+5=0 \\ 4x=-1 & & x=-5 \\ x=-\frac{1}{4} & & \end{array}$$

The rational zeros are 1, $-\frac{1}{4}$, and -5.

12. The potential numerators are ± 1 and ± 3 (factors of the constant term).

The potential denominators are ± 1 and ± 2 (factors of the initial coefficient).

The potential rational zeros are ± 1, ± 3, $\pm \frac{1}{2}$, and $\pm \frac{3}{2}$.

1	2	−9	11	−1	−3
		2	−11	22	−23
	2	−11	22	−23	20

Since the remainder is not 0, then $x+1$ is not a factor and -1 is not a zero.

−1	2	−9	11	−1	−3
		−2	7	−4	−3
	2	−7	4	3	0

Since the remainder is 0, then $x-1$ is a factor and 1 is a zero.

$$P(x)=(x-1)\left(2x^3-7x^2+4x+3\right)$$

The remaining factors and zeros must be obtained from the polynomial $2x^3-7x^2+4x+3$.

3	2	−7	4	3
		6	−39	129
	2	−13	43	−126

Since the remainder is not 0, then -3 is not a zero.

−3	2	−7	4	3
		−6	3	−3
	2	−1	1	6

Since the remainder is not 0, then 3 is not a zero.

$\frac{1}{2}$	2	−7	4	3
		1	−4	4
	2	−8	8	−1

Since the remainder is not 0, then $-\frac{1}{2}$ is not a zero.

$-\frac{1}{2}$	2	−7	4	3
		−1	3	$-\frac{1}{2}$
	2	−6	1	$3\frac{1}{2}$

Since the remainder is not 0, then $\frac{1}{2}$ is not a zero.

$\frac{3}{2}$	2	−7	4	3
		3	−15	$28\frac{1}{2}$
	2	−10	19	$-25\frac{1}{2}$

Since the remainder is not 0, then $-\frac{3}{2}$ is not a zero.

$-\frac{3}{2}$	2	−7	4	3
		−3	6	3
	2	−4	−2	0

Since the remainder is 0, $x-\frac{3}{2}$ is a factor and $\frac{3}{2}$ is a zero.

$P(x)=(x-1)\left(x-\frac{3}{2}\right)\left(2x^2-4x-2\right)$. Since $\frac{3}{2}$ is a rational number, $P(x)$ must be changed to the integral form.

$$P(x)=(x-1)\left(x-\frac{3}{2}\right)(2)\left(x^2-2x-1\right)$$
$$=(x-1)(2x-3)\left(x^2-2x-1\right)$$

Any remaining factors and zeros must be obtained from the polynomial x^2-2x-1. Since x^2-2x-1 cannot be factored by ordinary methods, the quadratic formula must be used.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad (\text{where } a=1,\ b=-2,\ \text{and } c=-1)$$
$$=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-1)}}{2(1)}$$
$$=\frac{2\pm\sqrt{4+4}}{2}$$
$$=\frac{2\pm2\sqrt{2}}{2}$$
$$=1\pm\sqrt{2}$$

The remaining zeros are $1+\sqrt{2}$ and $1-\sqrt{2}$. The factored form of $P(x)$ over the integers is

$P(x)=(x-1)(2x-3)\left(x^2-2x-1\right)$.

The real zeros of $P(x)$ are 1, $\frac{3}{2}$, $1+\sqrt{2}$, and $1-\sqrt{2}$.

13. Solving the equation $12x^3+17x^2+2x-3=0$ for its roots is similar to determining the zeros of

$P(x)=12x^3+17x^2+2x-3$.

The potential numerators are ±1 and ±3 (factors of the constant term).

The potential denominators are ±1, ±2, ±3, ±4, ±6, and ±12.

The potential rational zeros are ±1, ±3, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{1}{3}$, $\pm\frac{1}{4}$, $\pm\frac{3}{4}$, $\pm\frac{1}{6}$, and $\pm\frac{1}{12}$.

1	12	17	2	−3
		12	5	−3
	12	5	−3	0

Since the remainder is 0, then $x+1$ is a factor and -1 is a zero.

The factored form is $(x+1)\left(12x^2+5x-3\right)=0$.

The remaining zeros are zeros of the quotient polynomial $12x^2+5x-3$.

$$12x^2+5x-3=(3x-1)(4x+3)$$

Let $(3x-1)(4x+3)=0$.

$3x-1=0$ or $4x+3=0$

$3x=1$ $\qquad$ $4x=-3$

$x=\frac{1}{3}$ $\qquad$ $x=-\frac{3}{4}$

All the roots of $12x^3+17x^2+2x-3=0$ are -1, $\frac{1}{3}$, and $-\frac{3}{4}$.

14. A polynomial with zeros of 1, -2, and $\frac{1}{4}$ (multiplicity of two) is given by $P(x)=k(x-1)(x+2)(4x-1)(4x-1)$.

A remainder of 100 is produced when the divisor is $x+1$; thus, $P(-1)=100$.

$$P(-1)=k(-1-1)(-1+2)[(4)(-1)-1][(4)(-1)-1]$$
$$100=k(-2)(1)(-5)(-5)$$
$$100=-50k$$
$$-2=k$$

$$\therefore P(x)=-2(x-1)(x+2)(4x-1)(4x-1)$$
$$=-32x^4-16x^3+78x^2-34x+4$$

Enrichment

1. The potential zeros are ±1, ±2, ±3, and ±6.

$$P(1)=(1)^5-8(1)^3+6(1)^2+7(1)-6$$
$$=1-8+6+7-6$$
$$=0$$

Thus, 1 is a zero.

Since 1 is a zero, then start with -1 in the divisor of the synthetic division.

-1	1	0	-8	6	7	-6
		-1	-1	7	1	-6
	1	1	-7	-1	6	0

Since the remainder is 0, the zero is 1.

1	1	1	-7	-1	6
		1	0	-7	6
	1	0	-7	6	0

Since the remainder is 0, the zero is -1.

2	1	0	-7	6
		2	-4	-6
	1	-2	-3	12

Since the remainder is 12, then -2 is not a zero.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -7 & 6 \\ & & -2 & -4 & 6 \\ \hline & 1 & 2 & -3 & \boxed{0} \end{array}$$

Since the remainder is 0, then 2 is a zero.

$$Q(x) = x^2 + 2x - 3 = (x+3)(x-1)$$

Let $(x+3)(x-1) = 0$.

$x+3=0$ or $x-1=0$

$x=-3$ $\qquad$ $x=1$

The solution set for $P(x)$ is $x=-3$, $x=-1$, $x=1$, and $x=2$. $x=1$ appears twice.

2. a. The smallest constant k is 20 since both ± 4 and ± 5 are factors of this constant.

b. The potential integral zeros are ± 1, ± 2, ± 4, ± 5, ± 10, and ± 20.

$$\begin{aligned} P(1) &= (1)^4 + 4(1)^3 - 9(1)^2 - 16(1) + 20 \\ &= 1 + 4 - 9 - 16 + 20 \\ &= 0 \end{aligned}$$

Thus, 1 is a zero.

$$\begin{aligned} P(-1) &= (-1)^4 + 4(-1)^3 - 9(-1)^2 - 16(-1) + 20 \\ &= 1 - 4 - 9 + 16 + 20 \\ &\neq 0 \end{aligned}$$

Thus, -1 is not a zero.

$$\begin{aligned} P(2) &= (2)^4 + 4(2)^3 - 9(2)^2 - 16(2) + 20 \\ &= 16 + 32 - 36 - 32 + 20 \\ &= 0 \end{aligned}$$

Thus, 2 is a zero.

Since 1 is a zero, then $x-1$ is a factor.

$$\begin{array}{r|rrrrr} -1 & 1 & 4 & -9 & -16 & 20 \\ & & -1 & -5 & 4 & 20 \\ \hline & 1 & 5 & -4 & -20 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x-1$ is a factor and 1 is a zero. This checks with direct substitution.

Since 2 is a zero, then $x-2$ is a factor.

$$\begin{array}{r|rrrr} -2 & 1 & 5 & -4 & -20 \\ & & -2 & -14 & -20 \\ \hline & 1 & 7 & 10 & \boxed{0} \end{array}$$

Thus, $x-2$ is a factor and 2 is a zero since the remainder is 0. This checks with direct substitution.

Thus, $P(x)=(x-1)(x-2)\left(x^2+7x+10\right)$. The remaining zeros are from the polynomial $x^2+7x+10$.

$$x^2+7x+10=(x+2)(x+5)$$

Let $(x+2)(x+5)=0$.

$x+2=0$ or $x+5=0$

$x=-2$ $\quad$ $x=-5$

The actual integral zeros are 1, 2, -2, and -5.

3. $6x^4-2x^2=13x^3-7x-2$

Transpose all terms to the left side.

$$6x^4-13x^3-2x^2+7x+2=0$$

Solving this equation for its roots is similar to determining the zeros of $P(x)=6x^4-13x^3-2x^2+7x+2$.

The potential numerators are ±1 and ±2 (factors of the constant term).

The potential denominators are ±1, ±2, ±3, and ±6 (factors of the initial coefficient.

The potential rational zeros are ±1, ±2, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, and $\pm\frac{1}{6}$.

$$\begin{aligned}P(1)&=6(1)^4-13(1)^3-2(1)^2+7(1)+2\\&=6-13-2+7+2\\&=0\end{aligned}$$

Thus, 1 is a zero. Since 1 is a zero, then $x-1$ is a factor. Start with -1 in the divisor of the synthetic division method.

-1	6	-13	-2	7	2
		-6	7	9	2
	6	-7	-9	-2	0

Since the remainder is 0, then $x-1$ is a factor and 1 is a zero. This checks with direct substitution.

The factored form is $(x-1)\left(6x^3-7x^2-9x-2\right)=0$.

1	6	-7	-9	-2
		6	-13	4
	6	-13	4	-6

Since the remainder is not 0, then -1 is not a zero.

2	6	-7	-9	-2
		12	-38	58
	6	-19	29	-60

Since the remainder is not 0, then -2 is not a zero.

$$\begin{array}{r|cccc} -2 & 6 & -7 & -9 & -2 \\ & & -12 & -10 & -2 \\ \hline & 6 & 5 & 1 & 0 \end{array}$$

Since the remainder is 0, then $x-2$ is a factor and 2 is a zero.

The factored form is $(x-1)(x-2)\left(6x^2+5x+1\right)=0$.

The remaining zeros can be determined from the polynomial $6x^2+5x+1$.

$6x^2+5x+1=(3x+1)(2x+1)$

Let $(3x+1)(2x+1)=0$.

$3x+1=0$ or $2x+1=0$

$3x=-1$ $\quad$ $2x=-1$

$x=-\frac{1}{3}$ $\quad$ $x=-\frac{1}{2}$

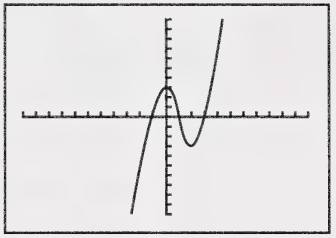
The roots of the equation are 1, 2, $-\frac{1}{3}$, and $-\frac{1}{2}$.

4. Every negative number with an odd number exponent is a negative number. The sum of negative numbers is a negative number.

5. Since x appears with even numbers as exponents, every nonzero real number replacement for x will yield a positive number and the sum of positive numbers is positive.

Section 4: Activity 1

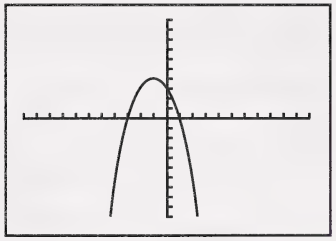
1. Graph X,θ,T x^y 3 − 3 X,θ,T x^y 2 − X,θ,T + 3 EXE



The y-intercept is 3.

The x-intercepts are -1, 1, and 3.

2. a. Graph − X,θ,T x^y 2 − 2 X,θ,T + 3 EXE

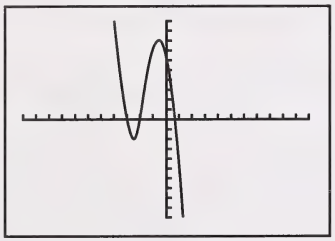


The y-intercept is 3.

The x-intercepts are -3 and 1.

b. Graph (−) 2 X,θ,T x^y 3 (−) 9 X,θ,T x^y

2 (−) 7 X,θ,T (+) 6 EXE



The y-intercept is 6.

The x-intercepts are -3, -2, and 0.5.

Section 4: Activity 2

1. a. $(+3)(-2)(+1)=-6$

The y-intercept is -6.

b. The zeros are -3, 2, and -1.

c. $(x)(x)(x)=x^3$

The degree of $P(x)$ is 3.

d. The initial coefficient is positive.

e. The graph begins by rising to the right in the third quadrant.

f. The graph ends by rising to the right in the first quadrant.

g. No roots have a multiplicity of more than one.

h. The polynomial has three real roots and no imaginary roots.

2. a. Let $x=0$; thus, the y-intercept is 0.

b. The zeros are 0, -2, 1, and 3.

c. $(-x)(x)(x)(x)=-x^4$

The degree of $P(x)$ is 4.

d. The initial coefficient is negative.

e. The graph begins by rising to the right in the third quadrant.

f. The graph ends by falling to the right in the fourth quadrant.

g. No roots have a multiplicity of more than one.

h. The polynomial has four real roots and no imaginary roots.

3. a. $(-1)(-2)(-2)(-4)=16$

The y-intercept is 16.

b. The zeros are 2 and 4.

c. The zero 2 has a multiplicity of two.
The zero 4 has a multiplicity of one.

d. $-(x)(x)(x)=-x^3$

The degree is 3.

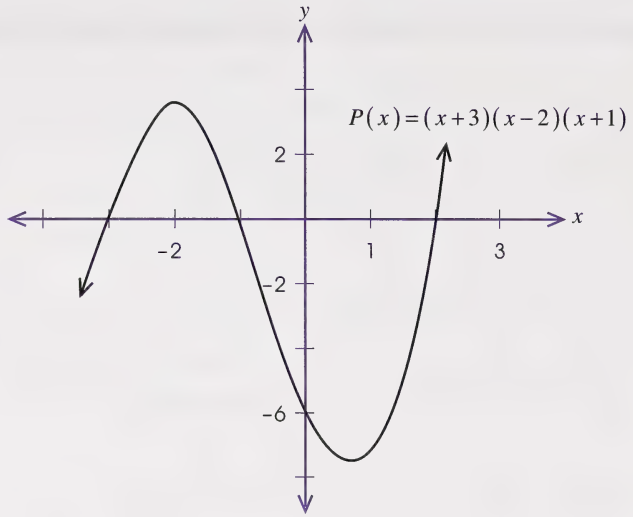
e. The initial coefficient is negative.

f. The graph begins by falling to the right in the second quadrant.

g. The graph ends by falling to the right in the fourth quadrant.

h. The polynomial has three real roots and no imaginary roots.

4.



5. a. $(1)(1)(-2)(-2)=4$

The y-intercept is 4.

b. The zeros are -1 and 2.

c. The zero -1 has a multiplicity of two.
The zero 2 has a multiplicity of two.

d. $(x)(x)(x)(x)=x^4$

The degree is 4.

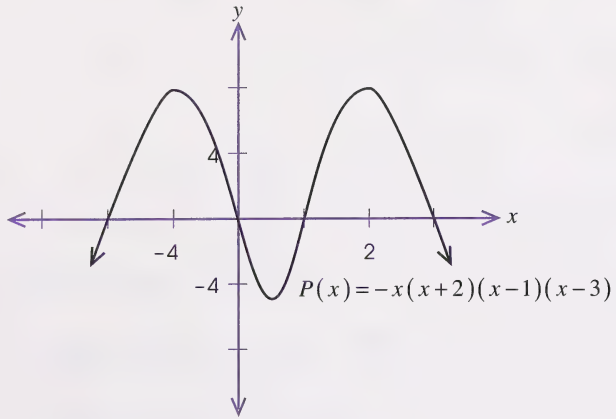
e. The initial coefficient is positive.

f. The graph begins by falling to the right in the second quadrant.

g. The graph ends by rising to the right in the first quadrant.

h. The polynomial has four real roots and no imaginary roots.

6.



7. **a.** Let $x = 0$; thus, the y-intercept is -6.

b. The factors are $(x+2)$ and $(x-3)$.

c. The zeros are -2 and 3.

d. The degree is 2.

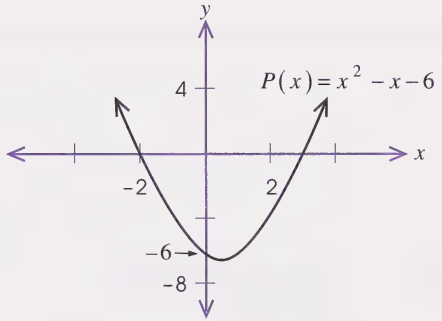
e. The initial coefficient is positive.

f. The graph begins by falling to the right in the second quadrant.

g. The graph ends by rising to the right in the first quadrant.

h. The polynomial has two real roots.

i.



8. **a.** **Step 1:** Let $x = 0$. Thus, the y-intercept is -6.

Step 2: The zeros of the polynomial must be determined.

The potential numerators are ± 1, ± 2, ± 3, and ± 6 (factors of the constant term).

The potential denominators are ± 1 and ± 3 (factors of the initial coefficient).

The potential rational zeros are ± 1, ± 2, ± 3, ± 6, $\pm \frac{1}{3}$, and $\pm \frac{2}{3}$.

Use synthetic division to determine the zeros.

$$\begin{array}{r|rrrr} 1 & -3 & 8 & 5 & -6 \\ & & -3 & 11 & -6 \\ \hline & -3 & 11 & -6 & 0 \end{array}$$

Since the remainder is 0, $x+1$ is a factor and -1 is a zero.

Thus, $P(x)=(x+1)\left(-3x^2+11x-6\right)$. The remaining two zeros must be determined from the quotient polynomial $-3x^2+11x-6$. Try factoring the quotient polynomial.

$$-3x^2+11x-6=(-3x+2)(x-3)$$

Let $(-3x+2)(x-3)=0$.

$-3x+2=0$ or $x-3=0$

$-3x=-2$ $\quad x=3$

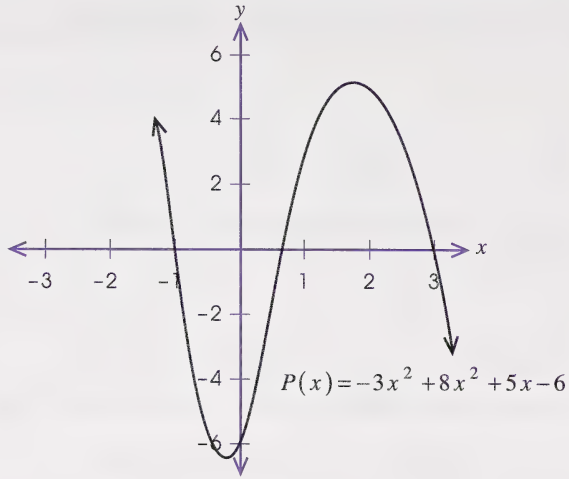
$x=\frac{2}{3}$

The zeros of the polynomial are -1, $\frac{2}{3}$, and 3.

Step 3: The polynomial has a degree of 3 and the initial coefficient a_n is negative. Therefore, the graph begins by falling to the right in the second quadrant and continues moving downward and to the right into the fourth quadrant.

Step 4: Each zero has a multiplicity of one. Thus, the curve crosses the x-axis at $(-1,0)$, $\left(\frac{2}{3},0\right)$, and $(3,0)$.

Step 5: The polynomial has three real zeros. The graph can now be sketched from the previous information.



Note: Since the graph is a sketch, the maximum and minimum values of the peaks and valleys will not agree with calculated values.

b. Step 1: Let $x=0$; thus, the y-intercept is 12.

Step 2: The zeros of the polynomial must be determined.

The potential numerators are ±1, ±2, ±3, ±4, ±6, and ±12.

The potential denominators are ±1, ±2, and ±4.

The potential rational zeros are ±1, ±2, ±3, ±4, ±6, ±12, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{1}{4}$, and $\pm\frac{3}{4}$.

Use synthetic division to calculate the zeros.

$$\begin{array}{r|rrrr} 1 & 4 & -3 & -16 & 12 \\ & & 4 & -7 & -9 \\ \hline & 4 & -7 & -9 & \boxed{21} \end{array}$$

Since the remainder is not 0, then -1 is not a zero.

$$\begin{array}{r|rrrr} -1 & 4 & -3 & -16 & 12 \\ & & -4 & -1 & 15 \\ \hline & 4 & 1 & -15 & \boxed{-3} \end{array}$$

Since the remainder is not 0, then 1 is not a zero.

$$\begin{array}{r|rrrr} 2 & 4 & -3 & -16 & 12 \\ & & 8 & -22 & 12 \\ \hline & 4 & -11 & 6 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x+2$ is a factor and -2 is a zero.

Thus, $P(x)=(x+2)\left(4x^2-11x+6\right)$.

The remaining two zeros must be determined from the quotient polynomial $4x^2-11x+6$. Try factoring the quotient polynomial.

$$4x^2-11x+6=0$$
$$(4x-3)(x-2)=0$$

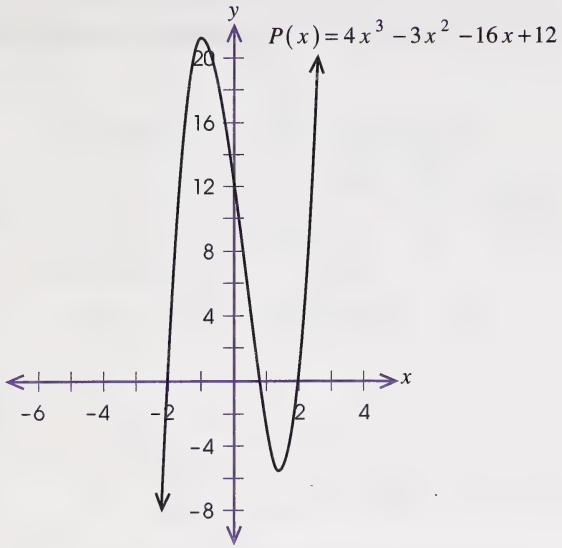
$$\therefore\ 4x-3=0 \quad \text{or} \quad x-2=0$$
$$4x=3 \qquad\qquad x=2$$
$$x=\frac{3}{4}$$

The zeros of the polynomial are -2, $\frac{3}{4}$, and 2.

Step 3: The polynomial has a degree of 3, and the initial coefficient a_n is positive. Thus, the graph begins by rising to the right in the third quadrant and ends by rising to the right in the first quadrant.

Step 4: Each zero has a multiplicity of one, which means the curve crosses the x-axis at $(-2,0)$, $\left(\frac{3}{4},0\right)$, and $(2,0)$.

Step 5: The polynomial has three real zeros. The graph can now be sketched from the previous information.



c. **Step 1:** Let $x=0$; thus, the y-intercept is 12.

Step 2: The zeros of the polynomial must be determined.

The potential numerators are ±1, ±2, ±3, ±4, ±6, and ±12.

The potential denominators are ±1 and ±2.

The potential rational zeros are ±1, ±2, ±3, ±4, ±6, ±12, $\pm\frac{1}{2}$, and $\pm\frac{3}{2}$.

Use synthetic division to calculate the zeros.

1	2	1	−17	−16	12
		2	−1	−16	0
	2	−1	−16	0	12

Since the remainder is not 0, then -1 is not a zero.

−1	2	1	−17	−16	12
		−2	−3	14	30
	2	3	−14	−30	−18

Since the remainder is not 0, then 1 is not a zero.

2	2	1	−17	−16	12
		4	−6	−22	12
	2	−3	−11	6	0

Since the remainder is 0, then $x+2$ is a factor and -2 is a zero.

Thus, $P(x)=(x+2)\left(2x^3-3x^2-11x+6\right)$.

Continue using synthetic division to find the zeros of $2x^3-3x^2-11x+6$.

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -11 & 6 \\ & & 4 & -14 & 6 \\ \hline & 2 & -7 & 3 & 0 \end{array}$$

Since the remainder is 0, then $x+2$ is a factor and -2 is a zero again.

Thus, $P(x)=(x+2)(x+2)\left(2x^2-7x+3\right)$.

The remaining two zeros must be determined from the quotient polynomial $2x^2-7x+3$. Try factoring the quotient polynomial $2x^2-7x+3$.

$$2x^2-7x+3=(2x-1)(x-3)$$

Let $(2x-1)(x-3)=0$.

$2x-1=0$ or $x-3=0$

$2x=1$ $\qquad x=3$

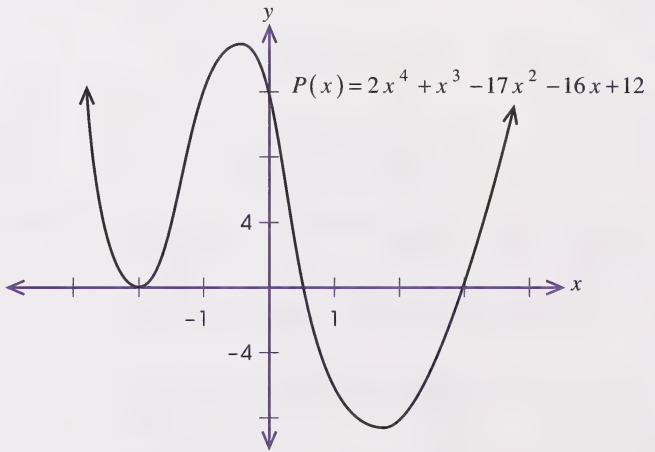
$x=\frac{1}{2}$

The zeros of the polynomial are $\frac{1}{2}$, 3, and -2 (multiplicity of two).

Step 3: The polynomial has a degree of 4, and the initial coefficient is positive. The graph begins by falling to the right in the second quadrant and ends by rising to the right in the first quadrant.

Step 4: The zero -2 has a multiplicity of two. Thus, the curve touches at $(-2,0)$. The zeros $\frac{1}{2}$ and 3 have a multiplicity of one. The curve will cross the x-axis at $\left(\frac{1}{2},0\right)$ and $(3,0)$.

Step 5: The polynomial has four real roots. Two of the roots are equal. The graph can now be drawn from the previous information.



d. Step 1: Let $x=0$. Thus, the y-intercept is -20.

Step 2: The zeros of the polynomial must be determined.

The potential numerators are ±1, ±2, ±4, ±5, ±10, and ±20.

The potential denominators are ±1 and ±2.

The potential rational zeros are ± 1, ± 2, ± 4, ± 5, ± 10, ± 20, $\pm \frac{1}{2}$, and $\pm \frac{5}{2}$.

Use synthetic division to calculate the zeros.

$$\begin{array}{r|rrrrr} 1 & -2 & -1 & 15 & 8 & -20 \\ & & -2 & 1 & 14 & -6 \\ \hline & -2 & 1 & 14 & -6 & \boxed{-14} \end{array}$$

Since the remainder is not 0, then -1 is not a zero.

$$\begin{array}{r|rrrrr} -1 & -2 & -1 & 15 & 8 & -20 \\ & & 2 & 3 & -12 & -20 \\ \hline & -2 & -3 & 12 & 20 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x-1$ is a factor and 1 is a zero.

Thus, $P(x)=(x-1)\left(-2x^3-3x^2+12x+20\right)$.

Use synthetic division to find the zeros of $-2x^3-3x^2+12x+20$.

$$\begin{array}{r|rrrr} -1 & -2 & -3 & 12 & 20 \\ & & 2 & 5 & -7 \\ \hline & -2 & -5 & 7 & \boxed{27} \end{array}$$

Since the remainder is not 0, then 1 is not a zero.

$$\begin{array}{r|rrrr} 2 & -2 & -3 & 12 & 20 \\ & & -4 & 2 & 20 \\ \hline & -2 & 1 & 10 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x+2$ is a factor and -2 is a zero.

Thus, $P(x)=(x-1)(x+2)\left(-2x^2+x+10\right)$.

The remaining two zeros must be determined from the quotient polynomial $-2x^2+x+10$. If this polynomial cannot be factored, then use the quadratic formula.

$$-2x^2+x+10=(-2x+5)(x+2)$$

Let $(-2x+5)(x+2)=0$.

$-2x+5=0$ or $x+2=0$

$-2x=-5$ $\qquad x=-2$

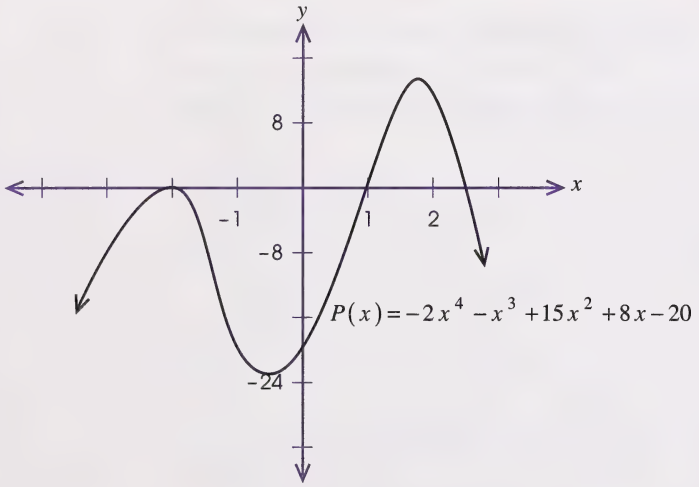
$x=\frac{5}{2}$

The zeros of the polynomial are 1, $\frac{5}{2}$, and -2 (multiplicity of two).

Step 3: The polynomial has a degree of 4, and the initial coefficient is negative. The graph begins by rising to the right in the third quadrant and ends by falling to the right in the fourth quadrant.

Step 4: The zero, -2, has a multiplicity of two. The curve touches the x-axis at $(-2,0)$. The zeros $\frac{5}{2}$ and 1 have a multiplicity of one. The curve will intersect the x-axis at $\left(\frac{5}{2},0\right)$ and $(1,0)$.

Step 5: The polynomial has four real zeros. Two of the zeros are equal. The sketch of the graph can now be drawn.



e. Step 1: To determine the y-intercept, let $x=0$. Thus, the y-intercept is 9.

Step 2: The zeros of the polynomial must be determined.

The potential numerators are ±1, ±3, and ±9.

The potential denominators are ±1, ±2, and ±4.

The potential rational zeros are ±1, ±3, ±9, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{9}{2}$, $\pm\frac{1}{4}$, $\pm\frac{3}{4}$, and $\pm\frac{9}{4}$.

Use synthetic division to calculate the zeros.

1	4	−4	−11	6	9
		4	−8	−3	9
	4	−8	−3	9	0

Since the remainder is 0, then $x+1$ is a factor and -1 is a zero.

Thus, $P(x)=(x+1)\left(4x^3-8x^2-3x+9\right)$.

Continue using synthetic division to find the zeros of $4x^3-8x^2-3x+9$.

1	4	−8	−3	9
		4	−12	9
	4	−12	9	0

Since the remainder is 0, then $x+1$ is a factor and -1 is a zero for the second time.

Thus, $P(x)=(x+1)(x+1)\left(4x^2-12x+9\right)$.

The remaining two zeros must be obtained from the quotient polynomial $4x^2 - 12x + 9$. Factor this polynomial to obtain the zeros. If the polynomial cannot be factored, then use the quadratic formula to obtain the zeros.

$$4x^2 - 12x + 9 = (2x - 3)(2x - 3)$$

Let $(2x - 3)(2x - 3) = 0$.

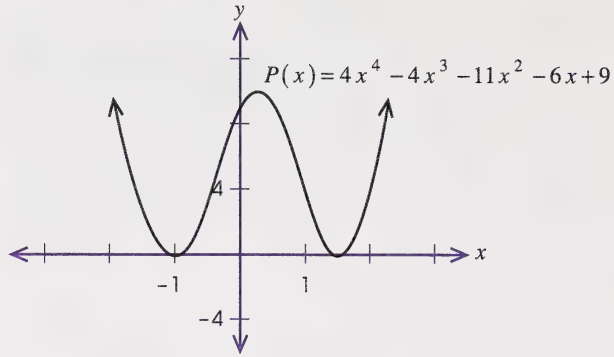
$$2x - 3 = 0 \quad \text{or} \quad 2x - 3 = 0$$
$$2x = 3 \qquad 2x = 3$$
$$x = \frac{3}{2} \qquad x = \frac{3}{2}$$

The zeros of the polynomial are -1 and $\frac{3}{2}$ (both have a multiplicity of two).

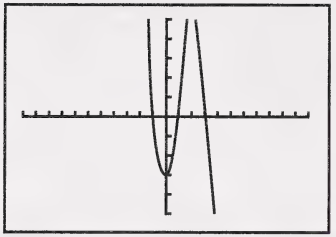
Step 3: The polynomial has a degree of 4, and the initial coefficient is positive. The graph begins by falling to the right in the second quadrant and ends by rising to the right in the first quadrant.

Step 4: The zeros -1 and $\frac{3}{2}$ each have a multiplicity of two. The curve touches the x-axis at $(-1, 0)$ and $\left(\frac{3}{2}, 0\right)$.

Step 5: The polynomial has four real zeros. The sketch of the graph is shown as follows:

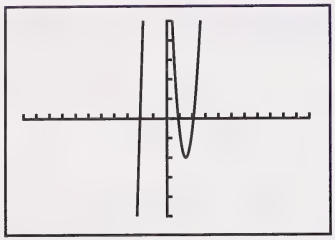


9. a.

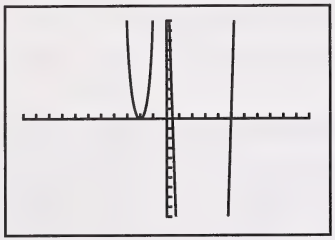


Note: The break in the graph at the top right side of the screen means the graph goes beyond the setting of the range. The graph is still one line even though it doesn't appear to be.

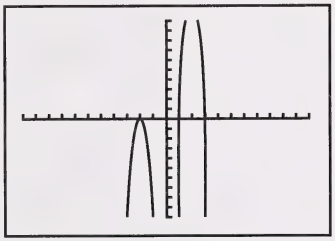
b.



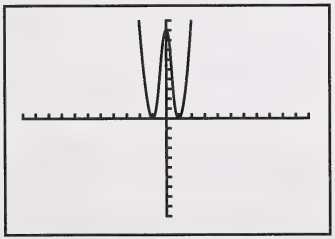
c.



d.



e.



Section 4: Activity 3

1. a. $P(x) = k(x-1)(3x-1)(4x+1)$

$P(-1) = k(-1-1)[3(-1)-1][4(-1)+1]$

$48 = k(-2)(-4)(-3)$ (since $P(-1) = 48$)

$48 = -24k$

$-2 = k$

$\therefore P(x) = -2(x-1)(3x-1)(4x+1)$

$= -24x^3 + 26x^2 - 2$

b. $P(x) = k(x+2)(2x-1)(2x+3)$

$P(1) = k(1+2)[(2)(1)-1][2(1)+3]$

$30 = k(3)(1)(5)$ (since $P(1) = 30$)

$30 = 15k$

$2 = k$

$\therefore P(x) = 2(x+2)(2x-1)(2x+3)$

$= 8x^3 + 24x^2 + 10x - 12$

2. a. $P(x)=k(x+1)(x-2)(2x+1)(3x-2)$

$P(1)=k(1+1)(1-2)[(2)(1)+1][3(1)-2]$

$6=k(2)(-1)(3)(1) \qquad (\text{since } P(1)=6)$

$6=-6k$

$k=-1$

$\therefore P(x)=-1(x+1)(x-2)(2x+1)(3x-2)$

$=-6x^4+7x^3+13x^2-4x-4$

b. $P(x)=k(x+2)(x+2)(3x+1)(3x+1)$

$P(1)=k(1+2)(1+2)[(3)(1)+1][3(1)+1]$

$288=k(3)(3)(4)(4) \qquad (\text{since } P(1)=288)$

$288=144k$

$2=k$

$\therefore P(x)=2(x+2)(x+2)(3x+1)(3x+1)$

$=18x^4+84x^3+122x^2+56x+8$

3. a. $P(x)=k(3x-1)(2x-3)(x+2)$

A y-intercept of -18 means $P(0)=-18$.

$\therefore -18=k(-1)(-3)(2)$

$-18=6k$

$-3=k$

$P(x)=-3(3x-1)(2x-3)(x+2)$

$=-18x^3-3x^2+57x-18$

b. $P(x)=k(2x-1)(x+1)(x+1)$

A y-intercept of 5 means $P(0)=5$.

$\therefore 5=k(-1)(1)(1)$

$5=-1k$

$-5=k$

$P(x)=-5(2x-1)(x+1)(x+1)$

$=-10x^3-15x^2+5$

4. a. $P(x)=k(x+1)(x-2)(3x-1)(2x+1)$

$=k\left(6x^4-5x^3-14x^2-1x+2\right)$

Since the initial coefficient must be -18, then $k=-3$.

$P(x)=-3\left(6x^4-5x^3-14x^2-1x+2\right)$

$=-18x^4+15x^3+42x^2+3x-6$

b. $P(x)=k(x-1)(x-1)(3x-2)(4x+1)$

$=k\left(12x^4-29x^3+20x^2-1x-2\right)$

Since the initial coefficient must be 24, then $k=2$.

$P(x)=2\left(12x^4-29x^3+20x^2-1x-2\right)$

$=24x^4-58x^3+40x^2-2x-4$

Section 4: Activity 4

1. a. The zero -2 has a multiplicity of one.
The zero -1 has a multiplicity of one.
The zero 2 has a multiplicity of one.
The zero 3 has a multiplicity of one.

The degree of the polynomial is the sum of the multiplicities. Thus, the degree of the polynomial is 4.

Since -2 is a zero, then $x+2$ is a factor.
Since -1 is a zero, then $x+1$ is a factor.
Since 2 is a zero, then $x-2$ is a factor.
Since 3 is a zero, then $x-3$ is a factor.

Because the graph begins by rising to the right in the third quadrant and ends by falling to the right in the fourth quadrant, the initial coefficient must be negative.

$P(x)=a(x+2)(x+1)(x-2)(x-3)$

Since $P(0)=-12$, solve for a.

$-12=a(0+2)(0+1)(0-2)(0-3)$

$-12=12a$

$a=-1$

$\therefore P(x)=-1(x+2)(x+1)(x-2)(x-3)$

$=-1\left(x^2+3x+2\right)\left(x^2-5x+6\right)$

$=-1\left(x^4-2x^3-7x^2+8x+12\right)$

$=-x^4+2x^3+7x^2-8x-12$

b. The zero -3 has a multiplicity of two.
The zero -1 has a multiplicity of one.
The zero 1 has a multiplicity of one.
The zero 2 has a multiplicity of one.

The degree of the polynomial is the sum of the multiplicities. Thus, the degree of the polynomial is 5.

Since the zero -3 has a multiplicity of two, then there are two factors: $(x+3)$ and $(x+3)$.
Since -1 is a zero, then $x+1$ is a factor.
Since 1 is a zero, then $x-1$ is a factor.
Since 2 is a zero, then $x-2$ is a factor.

Since the graph is of degree 5 and begins by falling to the right in the second quadrant and ends by falling to the right in the fourth quadrant, then the initial coefficient must be negative.

$$P(x)=a(x+3)(x+3)(x+1)(x-1)(x-2)$$

Since $P(0)=-36$, solve for a.

$$\begin{aligned}-36&=a(0+3)(0+3)(0+1)(0-1)(0-2)\\-36&=18a\\a&=-2\end{aligned}$$

$$\begin{aligned}\therefore\ P(x)&=-2(x+3)(x+3)(x+1)(x-1)(x-2)\\&=-2\left(x^3+7x^2+15x+9\right)\left(x^2-3x+2\right)\\&=-2\left(x^5+4x^4-4x^3-22x^2+3x+18\right)\\&=-2x^5-8x^4+8x^3+44x^2-6x-36\end{aligned}$$

c. The zero -2 has a multiplicity of two.
The zero 1 has a multiplicity of one.
The zero 3 has a multiplicity of one.

The degree of the polynomial is the sum of the multiplicities. Thus, the degree of the polynomial is 4.

Since the zero -2 has a multiplicity of two, then there are two factors: $(x+2)$ and $(x+2)$.

Since 1 is a zero, then $x-1$ is a factor.
Since 3 is a zero, then $x-3$ is a factor.

Since the graph is of degree 4 and begins by falling to the right in the second quadrant and ends by rising to the right in the first quadrant, then the initial coefficient must be positive.

$$P(x)=a(x+2)(x+2)(x-1)(x-3)$$

Since the y-intercept is not known then $P(0)$ is any number.

$\therefore\ a>0$

$$\begin{aligned}P(x)&=a(x+2)(x+2)(x-1)(x-3)\\&=a\left(x^2+4x+4\right)\left(x^2-4x+3\right)\\&=a\left(x^4-9x^2-4x+12\right)\end{aligned}$$

d. The zero -2 has a multiplicity of two.
The zero 1 has a multiplicity of three.

The degree of the polynomial is the sum of the multiplicities. Thus, the degree of the polynomial is 5.

Since the zero -2 has a multiplicity of two, then there are two factors: $(x+2)$ and $(x+2)$. Similarily, the zero 1 has a multiplicity of three. Thus, there are three factors: $(x-1)$, $(x-1)$, and $(x-1)$.

Since the graph is of degree 5 and begins by rising in the third quadrant and ends by rising in the first quadrant, then the initial coefficient must be positive.

$$P(x) = a(x+2)(x+2)(x-1)(x-1)(x-1)$$

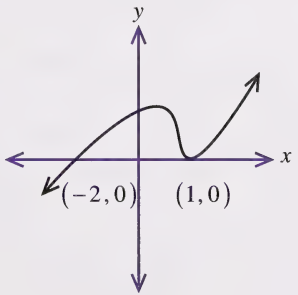
Since $P(0) = -16$, solve for a.

$$-16 = a(0+2)(0+2)(0-1)(0-1)(0-1)$$
$$-16 = -4a$$
$$a = 4$$

$$\therefore P(x) = 4(x+2)(x+2)(x-1)(x-1)(x-1)$$
$$= 4\left(x^2+4x+4\right)\left(x^3-3x^2+3x-1\right)$$
$$= 4x^5+4x^4-20x^3-4x^2+32x-16$$

2. The zero -2 has a multiplicity of one. The factor is $x+2$. The zero 1 has a multiplicity of two. The factors are $(x-1)$ and $(x-1)$.

The polynomial has a degree of 3. In order to have a y-intercept of 2, the graph must begin by rising to the right in the third quadrant and end by rising to the right in the first quadrant. This means the initial coefficient must be positive.



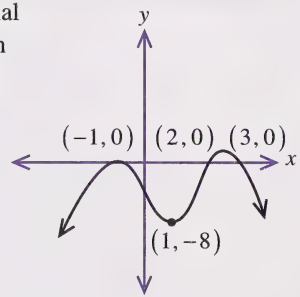
$$P(x) = k(x+2)(x-1)(x-1)$$

Since $P(0) = 2$, solve for k.

$$2 = k(0+2)(0-1)(0-1)$$
$$2 = 2k$$
$$k = 1$$

$$\therefore P(x) = 1(x+2)(x-1)(x-1)$$
$$= x^3-3x+2$$

3. The zero -1 has a multiplicity of two; thus, the factors are $(x+1)$ and $(x+1)$. The zero 2 has a multiplicity of one; thus, the factor is $(x-2)$. The zero 3 has a multiplicity of one; thus, the factor is $(x-3)$. The polynomial has a degree of 4 and passes through the point $(1,-8)$. In order to pass through point $(1,-8)$, the graph must begin by rising to the right in the third quadrant and end by falling to the right in the fourth quadrant. This means the initial coefficient must be negative.



$$P(x) = k(x+1)^2(x-2)(x-3)$$

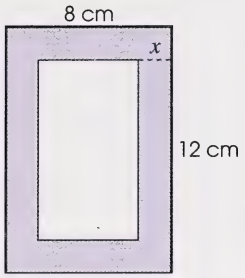
Since $P(1) = -8$, solve for k.

$$-8 = k(1+1)^2(1-2)(1-3)$$
$$-8 = 8k$$
$$k = -1$$

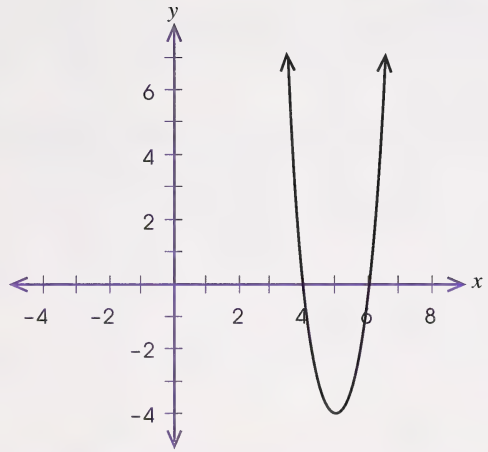
$$\therefore P(x) = -1(x+1)(x+1)(x-2)(x-3)$$
$$= -1\left(x^2+2x+1\right)\left(x^2-5x+6\right)$$
$$= -1\left(x^4-3x^3-3x^2+7x+6\right)$$
$$= -x^4+3x^3+3x^2-7x-6$$

4. a. $y = (12-2x)(8-2x)$

$= 4x^2 - 40x + 96$



b.



c. The width of the frame plus mirror is 8 cm. If $x = 4$ cm, there will be no room for the mirror.

d. $4x^2 - 40x + 96 = 32$

$4x^2 - 40x + 64 = 0$

$x^2 - 10x + 16 = 0$

$(x-2)(x-8) = 0$

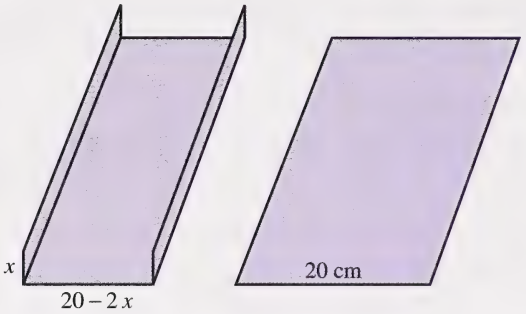
$x - 2 = 0$ or $x - 8 = 0$

$x = 2$ $\quad$ $x = 8$ (impossible)

The width of the frame is 2 cm.

e. The domain is $0 < x < 5$.

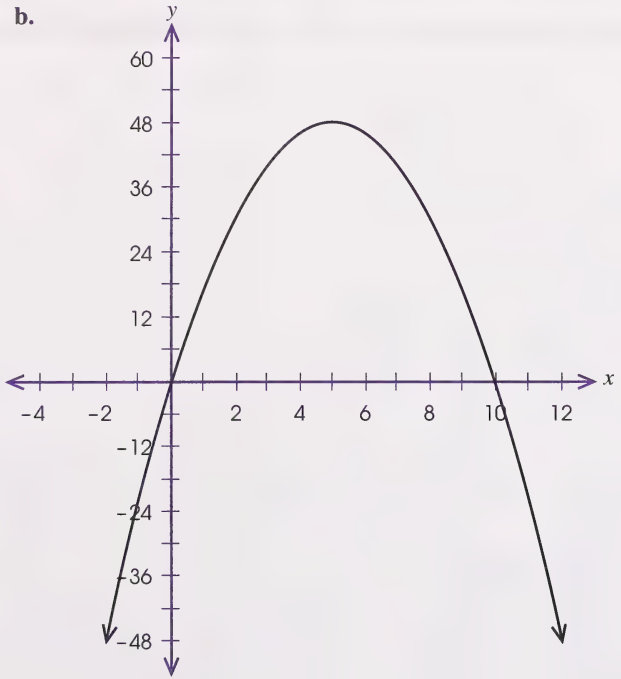
5. a.



$y = x(20-2x)$

$= -2x^2 + 20x$

b.

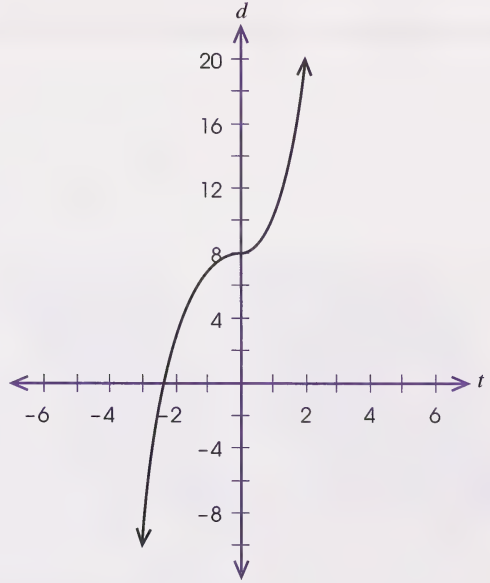


c. Yes, the domain is $0 < x < 10$.

d. It increases to a maximum and then decreases.

e. Either no metal has been turned up or 10 cm has been turned up. (The cross-section is zero.)

6. a.



b. The t-intercept represents when the object is at the origin.

c. At $t = 2$, $d = (2)^3 + (2)^2 + 8$

$= 8 + 4 + 8$

$= 20$

At $t = 2$, the object is 20 m away.

d. When $d = 158$, $158 = t^3 + t^2 + 8$

$$0 = t^3 + t^2 - 150$$

Let $F(t) = t^3 + t^2 - 150$.

$$\begin{aligned} F(5) &= (5)^3 + (5)^2 - 150 \\ &= 125 + 25 - 150 \\ &= 0 \end{aligned}$$

Therefore, the object is 158 m away when $t = 5$.

Section 4: Follow-up Activities

Extra Help

1. Ordered pairs of real numbers can be plotted on a rectangular coordinate system, which is also called a **Cartesian** coordinate system.

2. To draw a graph of a polynomial, use a **table of values**; then plot the points represented by each **ordered pair**.

3. The x-intercepts are the **zeros** of the polynomial.

4. To determine the y-intercept, let $\boldsymbol{x = 0}$.

5. The degree of a polynomial is at least **one more** than the number of changes of direction in the graph.

6. The zeros of a polynomial may be real or **imaginary**.

7. Real zeros occur at the points where the polynomial **crosses** or **touches** the x-axis.

8. The **multiplicity** of a zero is the number of times that the same zero occurs for a polynomial.

9. If the **multiplicity** of the zero is even, then the graph **touches** the x-axis. If the **multiplicity** is odd, then the graph **crosses** the x-axis.

10. $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$.
If $n = 3$ (or any odd integer) and a_n is positive, then the graph begins by **rising** in the **third** quadrant and ends by **rising** in the **first** quadrant.

11. If $n = 3$ (or any odd integer) and a_n is negative, then the graph begins by **falling** to the right in the **second** quadrant and ends by **falling** to the right in the **fourth** quadrant.

12. If $n = 4$ (or any even integer) and a_n is positive, then the graph begins by **falling** to the right in the **second** quadrant and ends by **rising** to the right in the **first** quadrant.

13. If $n = 4$ (or any even integer) and a_n is negative, then the graph begins by **rising** to the right in the **third** quadrant and ends by **falling** to the right in the **fourth** quadrant.

14. a. $-1(3)(2)(-1)(-1) = -6$

The y-intercept is -6.

b. The zeros are -3, -2, and 1.

c. The multiplicity of -3 is one.
The multiplicity of -2 is one.
The multiplicity of 1 is two.

d. $1+1+2=4$

The degree is 4.

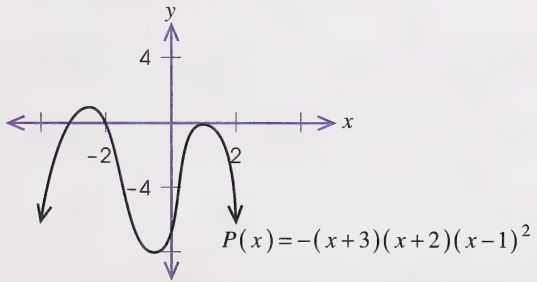
e. The initial coefficient is negative.

f. The graph begins by rising to the right in the third quadrant.

g. The graph ends by falling to the right in the fourth quadrant.

h. The polynomial has four real roots and no imaginary roots.

i.



15. Step 1: To obtain the y-intercept, let $x=0$; thus, the y-intercept is 12.

Step 2: The potential rational zeros are ±1, ±2, ±3, ±4, ±6, and ±12.

1	−1	−1	8	12
		−1	0	8
	−1	0	8	4

Since the remainder is not 0, then -1 is not a zero.

−1	−1	−1	8	12
		1	2	−6
	−1	−2	6	18

Since the remainder is not 0, then 1 is not a zero.

2	−1	−1	8	12
		−2	2	12
	−1	1	6	0

Since the remainder is 0, then $x+2$ is a factor and -2 is a zero.

$$P(x)=(x+2)\left(-x^2+x+6\right)$$

Now factor $-x^2+x+6$ to obtain the remaining two zeros.

$$\left(-x^2+x+6\right)=(-x-2)(x-3)$$

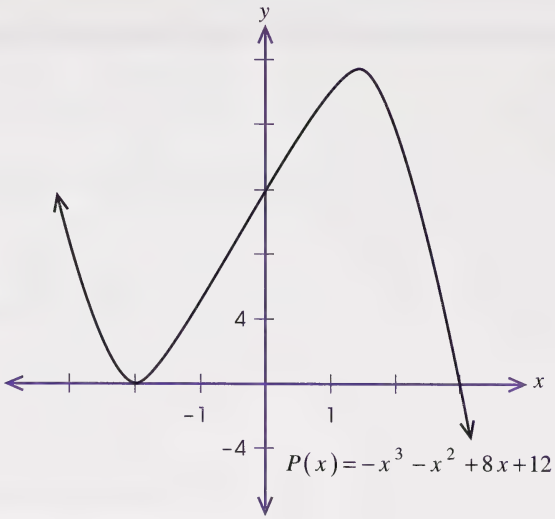
Let $(-x-2)(x-3)=0$.

$$-x-2=0 \quad \text{or} \quad x-3=0$$
$$-x=2 \qquad\qquad x=3$$
$$x=-2$$

Step 3: The polynomial has a degree of 3 and an initial coefficient of -1. The graph will begin by falling to the right in the second quadrant and end by falling to the right in the fourth quadrant.

Step 4: The real zero 3 has a multiplicity of one. The real zero -2 has a multiplicity of two. The graph will cross the x-axis at $(3,0)$ and touch the x-axis at $(-2,0)$.

Step 5: Use the y-intercept and the three real zeros to sketch the curve.



16. a. The graph crosses the x-axis at $(-2,0)$; thus, the zero -2 has a multiplicity of one. The graph also crosses the x-axis at $(-1,0)$; thus, -1 has a multiplicity of one. The graph has a triple root at $(2,0)$; thus, 2 has a multiplicity of three.

The factor for -2 is $x+2$.
The factor for -1 is $x+1$.
The factors for 2 are $(x-2)(x-2)(x-2)$.

b. The degree of the polynomial is the sum of the multiplicities. The degree is $1+1+3=5$.

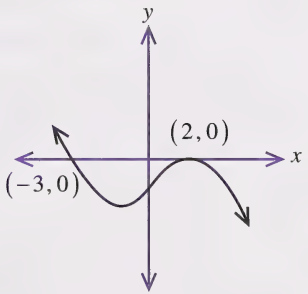
c. Since the graph of the polynomial starts rising in the third quadrant and the degree of the polynomial is 5, then the initial coefficient is positive.

$$P(x)=a(x+2)(x+1)(x-2)^3$$

Since $a=1$, the polynomial function, in factored form is $P(x)=(x+2)(x+1)(x-2)^3$.

d.
$$\begin{aligned}P(x)&=a(x+2)(x+1)(x-2)^3\\&=2(x+2)(x+1)(x-2)^3\\&=2\left(x^2+3x+2\right)\left(x^3-6x^2+12x-8\right)\\&=2x^5-6x^4-8x^3+32x^2-32\end{aligned}$$

17. Since the curve is tangent to the x-axis at $(2,0)$, then this means the zero 2 has a multiplicity of two. Since 2 is a zero, then $x-2$ is a factor. There will be two factors of $x-2$ since the multiplicity is two. The curve intersects the x-axis at $(-3,0)$. Since -3 is a zero, then $x+3$ is a factor. In order to pass through the point $(1,-4)$, the graph must begin by falling to the right in the second quadrant. This means the initial coefficient is negative.



Substitute $(1,-4)$ into the polynomial.

$$\begin{aligned}P(x)&=a(x+3)(x-2)(x-2)\\-4&=a(1+3)(1-2)(1-2)\\-4&=4a\\a&=-1\end{aligned}$$

$$\begin{aligned}P(x)&=-1(x-2)(x-2)(x+3)\\&=-1\left(x^2-4x+4\right)(x+3)\\&=-1\left(x^3-x^2-8x+12\right)\\&=-x^3+x^2+8x-12\end{aligned}$$

Enrichment

1. There are two valleys and one hill.
$(2+1)+1=4$
The degree is 4.
There are four real zeros.
The initial coefficient is positive.

2. There are two valleys and two hills.
$(2+2)+1=5$
The degree is 5.
There are five real zeros.
The initial coefficient is negative.

3. There are two valleys and two hills.
$(2+2)+1=5$
The degree is 5.
There is one real zero and four nonreal zeros.
The initial coefficient is negative.

4. There are two valleys and three hills.
$(2+3)+1=6$
The degree is 6.
There are four real zeros and two nonreal zeros.
The initial coefficient is negative.

5. There are two valleys and one hill.
$(2+1)+1=4$
The degree is 4.
There are two real zeros and two nonreal zeros.
The initial coefficient is positive.

6. There is one valley and two hills.
$(1+2)+1=4$
The degree is 4.
There are four real zeros (The negative zero has a multiplicity of three, and the positive zero has a multiplicity of one.)
The initial coefficient is negative.

7. If -3 is a zero, then $x-(-3)=x+3$ is a factor.

If $1+\sqrt{5}$ is a zero, then $x-\left(1+\sqrt{5}\right)$ is a factor.

If $1-\sqrt{5}$ is a zero, then $x-\left(1-\sqrt{5}\right)$ is a factor.

$$P(x)=(x+3)\left[x-\left(1+\sqrt{5}\right)\right]\left[x-\left(1-\sqrt{5}\right)\right]$$

The following shows $\left[x-\left(1+\sqrt{5}\right)\right]$ multiplied by $\left[x-\left(1-\sqrt{5}\right)\right]$.

$$\begin{array}{rr} & x-\left(1+\sqrt{5}\right) \\ & x-\left(1-\sqrt{5}\right) \\ \hline & -\left(1-\sqrt{5}\right)x-4 \\ x^2 & -\left(1+\sqrt{5}\right)x \\ \hline x^2 & -\left(1+\sqrt{5}\right)x-\left(1-\sqrt{5}\right)x-4 \end{array}$$

$$=x^2-x+\sqrt{5}x-x-\sqrt{5}x-4$$
$$=x^2-2x-4$$

$$\therefore P(x)=(x+3)\left(x^2-2x-4\right)$$
$$=x^3+x^2-10x-12$$

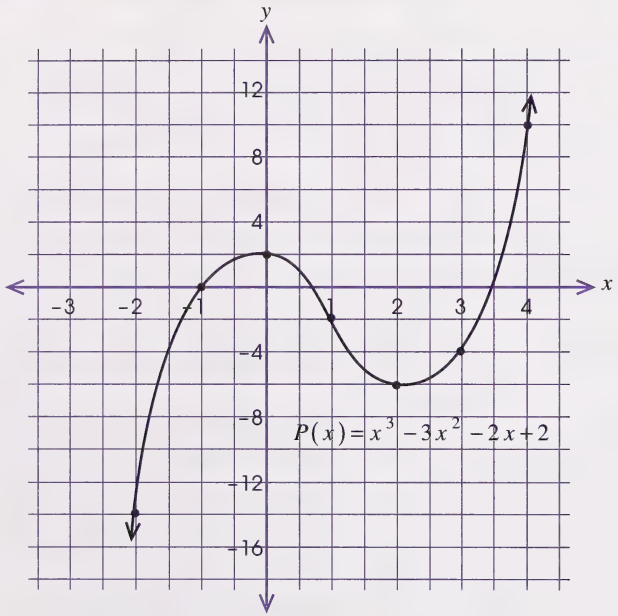
When $x=1$, $P(x)=(1)^3+1(1)^2-10(1)-12$
$=-20$

The polynomial has a value of -20 when x is 1, but the required value is -10. Since one-half of -20 equals -10, the polynomial must be multiplied by $\frac{1}{2}$.

$$P(x)=\frac{1}{2}\left(x^3+x^2-10x-12\right)$$
$$=\frac{1}{2}x^3+\frac{1}{2}x^2-5x-6$$

8.

x	-2	-1	0	1	2	3	4
$P(x)$	-14	0	2	-2	-6	-4	10



9. The potential rational zeros are ± 1 and ± 2.

$$\begin{array}{r|rrrr} 1 & 1 & -3 & -2 & 2 \\ & & 1 & -4 & 2 \\ \hline & 1 & -4 & 2 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x+1$ is a factor and -1 is a zero.

$$P(x)=(x+1)\left(x^2-4x+2\right)$$

From the graph, one irrational zero appears to be between 0.5 and 1. Try the value 0.5. (Use -0.5 as the divisor.)

$$\begin{array}{r|rrr} -0.5 & 1 & -4 & 2 \\ & & -0.5 & 1.75 \\ \hline & 1 & -3.5 & \boxed{0.25} \end{array}$$

Now, try the value of 0.6. (Use -0.6 for the divisor.)

$$\begin{array}{r|rrr} -0.6 & 1 & -4 & 2 \\ & & -0.6 & 2.04 \\ \hline & 1 & -3.4 & \boxed{-0.04} \end{array}$$

The irrational zero is between 0.5 and 0.6 since the remainder changes from $+0.25$ to -0.04. Since the remainder for 0.6 is closer to zero, then try values closer to 0.6. Try the value 0.58. (Use the divisor -0.58.)

$$\begin{array}{r|rrr} -0.58 & 1 & -4 & 2 \\ & & -0.58 & 1.9836 \\ \hline & 1 & -3.42 & \boxed{0.0164} \end{array}$$

Now try the value 0.59. (Use -0.59 as the divisor.)

-0.59	1	-4	2
		-0.59	2.0119
	1	-3.41	-0.0119

Since the remainder -0.0119 is closer to zero than the remainder 0.0164, then the irrational zero is 0.59 (to two decimal places).

From the graph, the other irrational zero appears to be between 3 and 3.5. Try the value 3.4.

-3.4	1	-4	2
		-3.4	2.04
	1	-0.6	-0.04

Now try the value 3.5.

-3.5	1	-4	2
		-3.5	1.75
	1	-0.5	0.25

The irrational zero is between 3.4 and 3.5 since the remainder changes from a negative value to a positive value. Since the remainder for 3.4 is closer to zero than the other remainder, then choose a value close to 3.4. Try the value 3.41.

-3.41	1	-4	2
		-3.41	2.0119
	1	-0.59	-0.0119

Now try the value 3.42.

-3.42	1	-4	2
		-3.42	1.9836
	1	-0.58	0.0164

Since the remainder -0.0119 is closer to zero than the remainder 0.0164, then the irrational zero is 3.41 (to two decimal places).

The polynomial has three real zeros: -1, 0.59 (to two decimal places), and 3.41 (to two decimal places).

NOTES

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